

# HAMILTONIAN STRUCTURES OF ISOMONODROMIC DEFORMATIONS ON MODULI SPACES OF PARABOLIC CONNECTIONS

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**ABSTRACT.** In this paper, we treat moduli spaces of parabolic connections. We take affine open coverings of the moduli spaces. We construct a Hamiltonian structure of an algebraic vector field determined by the isomonodromic deformation on each affine open set of the coverings.

## 1. INTRODUCTION

Let  $(C, \mathbf{t})$  ( $\mathbf{t} = (t_1, \dots, t_n)$ ) be an  $n$ -pointed smooth projective curve of genus  $g$  over  $\mathbb{C}$  where  $t_1, \dots, t_n$  are distinct points. We take a positive integer  $r$ , and take an element  $\boldsymbol{\nu} = (\nu_j^{(i)}) \in \mathbb{C}^{nr}$  such that  $\sum_{i,j} \nu_j^{(i)} = -d \in \mathbb{Z}$ . We say  $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$  a  $(\mathbf{t}, \boldsymbol{\nu})$ -parabolic connection of rank  $r$  if

- (1)  $E$  is a rank  $r$  algebraic vector bundle of degree  $d$  on  $C$ ,
- (2)  $\nabla: E \rightarrow E \otimes \Omega_C^1(t_1 + \dots + t_n)$  is a connection, and
- (3) for each  $t_i$ ,  $l_*^{(i)}$  is a filtration  $E|_{t_i} = l_0^{(i)} \supset l_1^{(i)} \supset \dots \supset l_r^{(i)} = 0$  such that  $\dim(l_j^{(i)}/l_{j+1}^{(i)}) = 1$  and  $(\text{res}_{t_i}(\nabla) - \nu_j^{(i)} \text{id}_{E|_{t_i}})(l_j^{(i)}) \subset l_{j+1}^{(i)}$  for  $j = 0, \dots, r-1$ .

Inaba–Iwasaki–Saito [5] (for general case Inaba [4]) introduces the  $\boldsymbol{\alpha}$ -stability for  $(\mathbf{t}, \boldsymbol{\nu})$ -parabolic connections, and constructs the moduli scheme of  $\boldsymbol{\alpha}$ -stable  $(\mathbf{t}, \boldsymbol{\nu})$ -parabolic connections of rank  $r$ , denoted by  $M_C^\alpha(\mathbf{t}, \boldsymbol{\nu})$ . Moreover, let  $T$  be a smooth algebraic scheme which is a certain covering of the moduli stack of  $n$ -pointed smooth projective curves of genus  $g$  over  $\mathbb{C}$  and take a universal family  $(\mathcal{C}, \tilde{t}_1, \dots, \tilde{t}_n)$  over  $T$ . Let  $N_r^{(n)}(d)$  be the set of  $\boldsymbol{\nu} = (\nu_j^{(i)}) \in \mathbb{C}^{nr}$  such that  $\sum_{i,j} \nu_j^{(i)} = -d \in \mathbb{Z}$ . Then we can construct a relative fine moduli scheme  $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d) \rightarrow T \times N_r^{(n)}(d)$  of  $\boldsymbol{\alpha}$ -stable parabolic connections of rank  $r$  and of degree  $d$ , which is smooth and quasi-projective [4, Theorem 2.1]. The moduli space  $M_C^\alpha(\mathbf{t}, \boldsymbol{\nu})$ , which is a fiber of  $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d) \rightarrow T \times N_r^{(n)}(d)$ , is equipped with a natural symplectic structure.

The moduli space  $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$  gives a geometric description of the differential equation determined by the isomonodromic deformation. We fix  $\boldsymbol{\nu}$ . We can regard  $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu \rightarrow T$  as a phase space of the differential equation determined by the isomonodromic deformation, and  $T$  as a space of time variables. A fiber of  $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu \rightarrow T$  becomes a space of initial conditions. In fact, for the case of  $C = \mathbb{P}^1$ ,  $r = 2$  and  $n = 4$ , these fibers coincide with the spaces of initial conditions for the Painlevé VI equation constructed by Okamoto [12] (see [6]). Inaba–Iwasaki–Saito [5] (for rank 2 and  $\mathbb{P}^1$  cases) and Inaba [4] (for general cases) show that the Riemann–Hilbert correspondence induces a proper surjective bimeromorphic morphism between the moduli space of  $\boldsymbol{\alpha}$ -stable parabolic connections and the moduli space of certain equivalent classes of representations of the fundamental group  $\pi_1(C \setminus \{t_1, \dots, t_n\}, *)$ . By this property of the Riemann–Hilbert correspondence, they show that the differential equation determined by the isomonodromic deformation satisfies the geometric Painlevé property (see [5] and [4]).

It is one of the important properties of the Painlevé equations that the Painlevé equations are written in (*non-autonomous*) *Hamiltonian systems* ([10], [11], [13], [16], [18]). The purpose of this paper is to give a Hamiltonian description of the vector field determined by the isomonodromic deformation on the moduli space  $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$ , which is a phase space. Hamiltonian description of the vector fields

determined by the isomonodromic deformations on moduli spaces of certain connections was essentially considered by Krichever [9] and Hurtubise [3]. (Wong [19] generalize those results for principal  $G$ -bundles cases). We apply their ideas to the moduli space  $M_{C/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$  of parabolic connections. Accordingly, we give an affine open covering of  $M_{C/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$ , and we give a Hamiltonian structure on each affine open set. Namely, we construct a two form  $\omega$  on  $M_{C/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$  such that the kernel  $\text{Ker}(\omega)$  implies the vector field determined by the isomonodromic deformation and  $\omega$  is the symplectic form fiberwise (Proposition 4.5). The two form  $\omega$  is considered in [7] and [8]. We define Hamiltonian functions  $H_i$   $i = 1, 2, \dots, \dim T$  on each affine open set (Definition 4.6). If we take good coordinates on  $M_{C/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$ , we obtain a Hamiltonian description of the vector field determined by the isomonodromic deformation (Corollary 4.8).

The organization of this paper is as follows. In Section 2, we recall the description of tangent spaces of  $M_C^\alpha(\mathbf{t}, \nu)$  by the hypercohomology of a certain complex, and the natural symplectic structure on  $M_C^\alpha(\mathbf{t}, \nu)$ . In Section 3, 3.2, we discuss a description of tangent spaces of  $M_{C/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$  by the hypercohomology of a certain complex. For the description, we recall the *Atiyah algebra* in 3.1. In 3.3, we describe the vector field determined by the isomonodromic deformation by the hypercohomology. In Section 4, we give a Hamiltonian description of the vector field determined by the isomonodromic deformation. In 4.1, we construct a relative initial connection  $\nabla_0$ . In 4.2, we define vector fields on each affine open set of  $M_{C/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$  associated to the relative initial connection  $\nabla_0$ . These vector fields are considered as vector fields associated to time variables. In 4.3, we give the two form  $\omega$  on  $M_{C/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$ , and define Hamiltonian functions on each affine open set of the moduli space  $M_{C/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$ . Finally, we obtain a Hamiltonian description of the vector field determined by the isomonodromic deformation on each affine open set.

## 2. PRELIMINARIES

**2.1. Moduli space of stable parabolic connections.** Let  $C$  be a smooth projective curve of genus  $g$ . We put

$$T_n := \{(t_1, \dots, t_n) \in C \times \dots \times C \mid t_i \neq t_j \text{ for } i \neq j\}$$

for a positive integer  $n$ . For integers  $d, r$  with  $r > 0$ , we put

$$N_r^{(n)}(d) := \left\{ (\nu_j^{(i)})_{0 \leq j \leq r-1}^{1 \leq i \leq n} \in \mathbb{C}^{nr} \mid d + \sum_{i,j} \nu_j^{(i)} = 0 \right\}.$$

Take members  $\mathbf{t} = (t_1, \dots, t_n) \in T_n$  and  $\nu = (\nu_j^{(i)})_{1 \leq i \leq n, 0 \leq j \leq r-1} \in N_r^{(n)}(d)$ .

**Definition 2.1.** We say  $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$  a  $(\mathbf{t}, \nu)$ -parabolic connection of rank  $r$  and degree  $d$  over  $C$  if

- (1)  $E$  is a rank  $r$  algebraic vector bundle on  $C$ ,
- (2)  $\nabla: E \rightarrow E \otimes \Omega_C^1(t_1 + \dots + t_n)$  is a connection, that is,  $\nabla$  is a homomorphism of sheaves satisfying  $\nabla(fa) = a \otimes df + f\nabla(a)$  for  $f \in \mathcal{O}_C$  and  $a \in E$ , and
- (3) for each  $t_i$ ,  $l_*^{(i)}$  is a filtration  $E|_{t_i} = l_0^{(i)} \supset l_1^{(i)} \supset \dots \supset l_r^{(i)} = 0$  such that  $\dim(l_j^{(i)}/l_{j+1}^{(i)}) = 1$  and  $(\text{res}_{t_i}(\nabla) - \nu_j^{(i)} \text{id}_{E|_{t_i}})(l_j^{(i)}) \subset l_{j+1}^{(i)}$  for  $j = 0, \dots, r-1$ .

**Remark 2.2.** We have

$$\deg E = \deg(\det(E)) = - \sum_{i=1}^n \text{res}_{t_i}(\nabla_{\det(E)}) = - \sum_{i=1}^n \sum_{j=0}^{r-1} \nu_j^{(i)} = d.$$

Take rational numbers

$$0 < \alpha_1^{(i)} < \alpha_2^{(i)} < \dots < \alpha_r^{(i)} < 1$$

for  $i = 1, \dots, n$  satisfying  $\alpha_j^{(i)} \neq \alpha_{j'}^{(i')}$  for  $(i, j) \neq (i', j')$ . We choose  $\alpha = (\alpha_j^{(i)})$  sufficiently generic.

**Definition 2.3.** A parabolic connection  $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$  is  $\alpha$ -stable (resp.  $\alpha$ -semistable) if for any proper nonzero subbundle  $F \subset E$  satisfying  $\nabla(F) \subset F \otimes \Omega_C^1(t_1 + \dots + t_n)$ , the inequality

$$\frac{\deg F + \sum_{i=1}^n \sum_{j=1}^r \alpha_j^{(i)} \dim((F|_{t_i} \cap l_{j-1}^{(i)})/(F|_{t_i} \cap l_j^{(i)}))}{\text{rank } F} \underset{(\text{resp. } \leq)}{<} \frac{\deg E + \sum_{i=1}^n \sum_{j=1}^r \alpha_j^{(i)} \dim(l_{j-1}^{(i)}/l_j^{(i)})}{\text{rank } E}$$

holds.

Let  $T$  be a smooth algebraic scheme which is a certain covering of the moduli stack of  $n$ -pointed smooth projective curves of genus  $g$  over  $\mathbb{C}$  and take a universal family  $(\mathcal{C}, \tilde{t}_1, \dots, \tilde{t}_n)$  over  $T$ .

**Definition 2.4.** We denote the bull-back of  $\mathcal{C}$  and  $\tilde{\mathbf{t}}$  by the morphism  $T \times N_r^{(n)}(d) \rightarrow T$  by the same character  $\mathcal{C}$  and  $\tilde{\mathbf{t}} = \{\tilde{t}_1, \dots, \tilde{t}_n\}$ . Then  $D(\tilde{\mathbf{t}}) := \tilde{t}_1 + \dots + \tilde{t}_n$  becomes an effective Cartier divisor on  $\mathcal{C}$  flat over  $T \times N_r^{(n)}(d)$ . We also denote by  $\tilde{\nu}$  the pull-back of the universal family on  $N_r^{(n)}(d)$  by the morphism  $T \times N_r^{(n)}(d) \rightarrow T$ . We define a functor  $\mathcal{M}_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$  of the category of locally noetherian schemes over  $T \times N_r^{(n)}(d)$  to the category of sets by

$$\mathcal{M}_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)(S) := \left\{ (E, \nabla, \{l_j^{(i)}\}) \right\} / \sim,$$

for a locally noetherian scheme  $S$  over  $T \times N_r^{(n)}(d)$ , where

- (1)  $E$  is a rank  $r$  algebraic vector bundle on  $\mathcal{C}_S$ ,
- (2)  $\nabla: E \rightarrow E \otimes \Omega_{\mathcal{C}_S/S}^1(D(\tilde{\mathbf{t}})_S)$  is a relative connection,
- (3) for each  $(\tilde{t}_i)_S$ ,  $l_*^{(i)}$  is a filtration by subbundles  $E|_{(\tilde{t}_i)_S} = l_0^{(i)} \supset l_1^{(i)} \supset \dots \supset l_r^{(i)} = 0$  such that  $(\text{res}_{(\tilde{t}_i)_S}(\nabla) - (\tilde{\nu}_j^{(i)})_{S \text{ id}_{E|_{t_i}}})(l_j^{(i)}) \subset l_{j+1}^{(i)}$  for  $j = 0, \dots, r-1$ , and
- (4) for any geometric point  $s \in S$ ,  $\dim(l_j^{(i)}/l_{j+1}^{(i)}) \otimes k(s) = 1$  for any  $i, j$  and  $(E, \nabla, \{l_j^{(i)}\}) \otimes k(s)$  is  $\alpha$ -stable.

Here  $(E, \nabla, \{l_j^{(i)}\}) \sim (E', \nabla', \{l'_j{}^{(i)}\})$  if there exists a line bundle  $\mathcal{L}$  on  $S$  and an isomorphism  $\sigma: E \xrightarrow{\sim} E' \otimes \mathcal{L}$  such that  $\sigma|_{t_i}(l_j^{(i)}) = l'_j{}^{(i)} \otimes \mathcal{L}$  for any  $i, j$  and the diagram

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & E \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}})) \\ \downarrow \sigma & & \downarrow \sigma \otimes \text{id} \\ E' \otimes \mathcal{L} & \xrightarrow{\nabla' \otimes \mathcal{L}} & E' \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}})) \otimes \mathcal{L} \end{array}$$

commutes.

**Theorem 2.5** ([4, Theorem 2.1]). *For the moduli functor  $\mathcal{M}_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ , there exists a fine moduli scheme*

$$M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d) \longrightarrow T \times N_r^{(n)}(d)$$

*of  $\alpha$ -stable parabolic connections of rank  $r$  and degree  $d$ , which is smooth and quasi-projective. The fiber  $M_{\mathcal{C}_x}^\alpha(\tilde{\mathbf{t}}_x, \nu)$  over  $(x, \nu) \in T \times N_r^{(n)}(d)$  is the moduli space of  $\alpha$ -stable  $(\tilde{\mathbf{t}}_x, \nu)$ -parabolic connections whose dimension is*

$$2r^2(g-1) + nr(r-1) + 2$$

*if it is non-empty.*

**2.2. Infinitesimal deformations.** We recall the description of the relative tangent sheaf  $\Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/T \times N_r^{(n)}(d)}$  by the hypercohomology of a certain complex ([4, the proof of Theorem 2.1]). Let  $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$  be a universal family on  $\mathcal{C} \times_T M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ . First, we define a complex  $\mathcal{F}^\bullet$  by

$$(1) \quad \begin{aligned} \mathcal{F}^0 &:= \left\{ s \in \mathcal{E}nd(\tilde{E}) \mid s|_{\tilde{t}_i \times M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)}(\tilde{l}_j^{(i)}) \subset \tilde{l}_j^{(i)} \text{ for any } i, j \right\} \\ \mathcal{F}^1 &:= \left\{ s \in \mathcal{E}nd(\tilde{E}) \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}})) \mid \text{res}_{\tilde{t}_i \times M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)}(s)(\tilde{l}_j^{(i)}) \subset \tilde{l}_{j+1}^{(i)} \text{ for any } i, j \right\} \\ \nabla_{\mathcal{F}^\bullet}: \mathcal{F}^0 &\longrightarrow \mathcal{F}^1; \quad \nabla_{\mathcal{F}^\bullet}(s) = \tilde{\nabla} \circ s - s \circ \tilde{\nabla}. \end{aligned}$$

Second, we take an affine open set  $M \subset M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$  and an affine open covering  $\mathcal{C}_M = \bigcup_\alpha U_\alpha$  such that  $\tilde{E}|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}^{\oplus r}$  for any  $\alpha$ ,  $\#\{i \mid \tilde{t}_i|_{\mathcal{C}_M} \cap U_\alpha \neq \emptyset\} \leq 1$  for any  $\alpha$  and  $\#\{\alpha \mid \tilde{t}_i|_{\mathcal{C}_M} \cap U_\alpha \neq \emptyset\} \leq 1$  for any  $i$ . Take a relative tangent vector field  $v \in \Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/T \times N_r^{(n)}(d)}(M)$ . The field  $v$  corresponds to a member  $(E_\epsilon, \nabla_\epsilon, \{(l_\epsilon)_j^{(i)}\}) \in M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)(\text{Spec } \mathcal{O}_M[\epsilon])$  such that  $(E_\epsilon, \nabla_\epsilon, \{(l_\epsilon)_j^{(i)}\}) \otimes \mathcal{O}_M[\epsilon]/(\epsilon) \cong (\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})|_{\mathcal{C}_M}$ , where  $\mathcal{O}_M[\epsilon] = \mathcal{O}_M[t]/(t^2)$ . There is an isomorphism

$$(2) \quad \varphi_\alpha: E_\epsilon|_{U_\alpha \times \text{Spec } \mathcal{O}_M[\epsilon]} \xrightarrow{\sim} \mathcal{O}_{U_\alpha \times \text{Spec } \mathcal{O}_M[\epsilon]}^{\oplus r} \xrightarrow{\sim} \tilde{E}|_{U_\alpha} \otimes \mathcal{O}_M[\epsilon]$$

such that  $\varphi_\alpha \otimes \mathcal{O}_M[\epsilon]/(\epsilon): E_\epsilon \otimes \mathcal{O}_M[\epsilon]/(\epsilon)|_{U_\alpha} \xrightarrow{\sim} \tilde{E}|_{U_\alpha} \otimes \mathcal{O}_M[\epsilon]/(\epsilon) = \tilde{E}|_{U_\alpha}$  is the given isomorphism and that  $\varphi_\alpha|_{\tilde{t}_i \times \mathcal{O}_M[\epsilon]}((l_\epsilon)_j^{(i)}) = \tilde{l}_j^{(i)}|_{U_\alpha \times \text{Spec } \mathcal{O}_M[\epsilon]}$  if  $\tilde{t}_i|_{\mathcal{C}_M} \cap U_\alpha \neq \emptyset$ . We put

$$(3) \quad \begin{aligned} u_{\alpha\beta} &:= \varphi_\alpha \circ \varphi_\beta^{-1} - \text{id}_{\tilde{E}|_{U_{\alpha\beta} \times \text{Spec } \mathcal{O}_M[\epsilon]}}, \\ v_\alpha &:= (\varphi_\alpha \times \text{id}) \circ \nabla_\epsilon|_{U_\alpha \times \text{Spec } \mathcal{O}_M[\epsilon]} \circ \varphi_\alpha^{-1} - \tilde{\nabla}|_{U_\alpha \times \text{Spec } \mathcal{O}_M[\epsilon]}. \end{aligned}$$

Then  $\{u_{\alpha\beta}\} \in C^1((\epsilon) \otimes \mathcal{F}_M^0)$ ,  $\{v_\alpha\} \in C^0((\epsilon) \otimes \mathcal{F}_M^1)$  and

$$d\{u_{\alpha\beta}\} = \{u_{\beta\gamma} - u_{\alpha\gamma} + u_{\alpha\beta}\} = 0; \quad \nabla_{\mathcal{F}^\bullet}\{u_{\alpha\beta}\} = \{v_\beta - v_\alpha\} = d\{v_\alpha\}.$$

So  $[(\{u_{\alpha\beta}\}, \{v_\alpha\})]$  determines an element  $\sigma_M(v)$  of  $\mathbf{H}^1(\mathcal{F}_M^\bullet)$ . We can check that  $v \mapsto \sigma_M(v)$  determines an isomorphism

$$\sigma_M: \Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/T \times N_r^{(n)}(d)}(M) \xrightarrow{\sim} \mathbf{H}^1(\mathcal{F}_M^\bullet); \quad v \longmapsto \sigma_M(v).$$

The isomorphism  $\sigma_M$  induces a canonical isomorphism

$$(4) \quad \sigma: \Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/T \times N_r^{(n)}(d)} \xrightarrow{\sim} \mathbf{R}^1(\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)})_*(\mathcal{F}^\bullet),$$

where  $\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)}: \mathcal{C}_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)} \rightarrow M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$  is the natural morphism.

**2.3. Symplectic structure.** For each affine open subset  $U \subset M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ , we define a pairing

$$(5) \quad \begin{aligned} \mathbf{H}^1(\mathcal{C} \times_T U, \mathcal{F}_U^\bullet) \otimes \mathbf{H}^1(\mathcal{C} \times_T U, \mathcal{F}_U^\bullet) &\longrightarrow \mathbf{H}^2(\mathcal{C} \times_T U, \Omega_{\mathcal{C} \times_T U/U}^\bullet) \cong H^0(\mathcal{O}_U) \\ [(\{u_{\alpha,\beta}\}, \{v_\alpha\})] \otimes [(\{u'_{\alpha,\beta}\}, \{v'_\alpha\})] &\longmapsto [(\{\text{Tr}(u_{\alpha,\beta} \circ u'_{\beta,\gamma})\}, -\{\text{Tr}(u_{\alpha,\beta} \circ v'_\beta) - \text{Tr}(v_\alpha \circ u'_{\alpha,\beta})\})] \end{aligned}$$

where we consider in Čech cohomology with respect to an affine open covering  $\{U_\alpha\}$  of  $\mathcal{C} \times_T U$ ,  $\{u_{\alpha,\beta}\} \in C^1(\mathcal{F}^0)$ ,  $\{v_\alpha\} \in C^0(\mathcal{F}^1)$  and so on. This pairing determines a pairing

$$(6) \quad \omega: \mathbf{R}^1(\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)})_*(\mathcal{F}^\bullet) \otimes \mathbf{R}^1(\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)})_*(\mathcal{F}^\bullet) \longrightarrow \mathcal{O}_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)}.$$

This pairing is a nondegenerate relative 2-form, which follows from the Serre duality:

$$(7) \quad \begin{array}{ccccccccc} H^0(\mathcal{F}_x^0) & \longrightarrow & H^0(\mathcal{F}_x^1) & \longrightarrow & \mathbf{H}^1(\mathcal{F}_x^\bullet) & \longrightarrow & H^1(\mathcal{F}_x^0) & \longrightarrow & H^1(\mathcal{F}_x^1) \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ H^1(\mathcal{F}_x^1)^\vee & \longrightarrow & H^1(\mathcal{F}_x^0)^\vee & \longrightarrow & \mathbf{H}^1(\mathcal{F}_x^\bullet)^\vee & \longrightarrow & H^0(\mathcal{F}_x^1)^\vee & \longrightarrow & H^0(\mathcal{F}_x^0)^\vee \end{array}$$

for any point  $x \in M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ , and we have  $d\omega = 0$  (see [4, Proposition 7.3]).

**2.4. Elementary transform of parabolic connections.** We recall the elementary transform of parabolic connections [4, Section 3]. Let  $T$  be a connected noetherian scheme and  $\pi_T: \mathcal{C} \rightarrow T$  be a smooth projective morphism whose geometric fibers are curves of genus  $g$ . Let  $\mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)$  be the functor of the category of locally noetherian schemes over  $T$  to the category of sets defined by

$$\mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)(S) := \left\{ (E, \nabla, \{l_j^{(i)}\}) \right\} / \sim,$$

for a locally noetherian scheme  $S$  over  $T$ , where

- (1)  $E$  is a rank  $r$  algebraic vector bundle on  $\mathcal{C}_S$ ,
- (2)  $\nabla: E \rightarrow E \otimes \Omega_{\mathcal{C}_S/S}^1(D(\tilde{\mathbf{t}})_S)$  is a relative connection,
- (3) for each  $(\tilde{t}_i)_S$ ,  $l_*^{(i)}$  is a filtration by subbundles  $E|_{(\tilde{t}_i)_S} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_r^{(i)} = 0$  such that  $(\text{res}_{(\tilde{t}_i)_S}(\nabla) - (\tilde{\nu}_j^{(i)})_{\text{sid}_{E|_{t_i}}})(l_j^{(i)}) \subset l_{j+1}^{(i)}$  for  $j = 0, \dots, r-1$ , and
- (4) for any geometric point  $s \in S$ ,  $\dim(l_j^{(i)}/l_{j+1}^{(i)}) \otimes k(s) = 1$  for any  $i, j$ .

Here  $(E, \nabla, \{l_j^{(i)}\}) \sim (E', \nabla', \{l'_j{}^{(i)}\})$  if there exists a line bundle  $\mathcal{L}$  on  $S$  such that  $(E, \nabla, \{l_j^{(i)}\}) \cong (E', \nabla', \{l'_j{}^{(i)}\}) \otimes \mathcal{L}$ .

For a locally noetherian scheme  $S$  over  $T$ , take any member  $(E, \nabla, \{l_j^{(i)}\}) \in \mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)(S)$ . For  $(p, q)$  with  $1 \leq p \leq n$  and  $0 \leq q \leq r$ , we put

$$E' := \ker \left( E \longrightarrow (E|_{(\tilde{t}_p)_S})/l_q^{(p)} \right).$$

Then  $\nabla$  induces a relative connection

$$\nabla': E' \longrightarrow E' \otimes \Omega_{\mathcal{C}_S/S}^1((\tilde{t}_1)_S + \cdots + (\tilde{t}_n)_S)$$

such that

$$\begin{array}{ccc} E' & \xrightarrow{\nabla'} & E' \otimes \Omega_{\mathcal{C}_S/S}^1((\tilde{t}_1)_S + \cdots + (\tilde{t}_n)_S) \\ \downarrow & & \downarrow \\ E & \xrightarrow{\nabla} & E \otimes \Omega_{\mathcal{C}_S/S}^1((\tilde{t}_1)_S + \cdots + (\tilde{t}_n)_S) \end{array}$$

is commutative. Moreover, we can determine a parabolic structure  $\{(l')_j^{(i)}\}$  of  $(E', \nabla')$  naturally (see [4, Section 3]). Then  $(E', \nabla', \{(l')_j^{(i)}\})$  becomes a member of  $\mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d-q)(S)$ . We say  $(E', \nabla', \{(l')_j^{(i)}\})$  the *elementary transform* of  $(E, \nabla, \{l_j^{(i)}\})$  along  $(\tilde{t}_p)_S$  by  $l_q^{(p)}$ . The elementary transform induces an isomorphism

$$\begin{aligned} \text{Elm}_q^{(p)}: \mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) &\longrightarrow \mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d-q) \\ (E, \nabla, \{l_j^{(i)}\}) &\longmapsto (E', \nabla', \{(l')_j^{(i)}\}) \end{aligned}$$

of functors.

### 3. ISOMONODROMIC DEFORMATION

In this section, we consider description of the vector field determined by the isomonodromic deformation by a Čech cohomology. In 3.1, we recall the Atiyah algebra. By the Atiyah algebra, we obtain descriptions of first-order deformations of pairs of curves and vector bundles. In 3.2, we show that the relative tangent sheaf of  $\Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/N_r^{(n)}(d)}$  is isomorphic to the hypercohomology of a certain complex using the Atiyah algebra. In 3.3, we consider the integrable deformations of parabolic connections when the  $n$ -pointed curves deform. The integrable deformations of parabolic connections imply the isomonodromic deformations of the corresponding relative parabolic connections. First, we show the existence of integrable deformations of parabolic connections as in [4, Section 8]. The obstructions of the existence are in the second hypercohomology of a certain complex. Second, we show that the integrable deformations of parabolic connections are determined by the Kodaira–Spencer classes corresponding to deformations

of  $n$ -pointed curves. Finally, we obtain a description of the vector field determined by the isomonodromic deformation by a Čech cohomology.

**3.1. Atiyah algebra.** We recall the *Atiyah algebra*. (For details, for example see [1]). Let  $C$  be a smooth projective curve, and  $\Theta_C$  be the tangent sheaf. Let  $E$  be a vector bundle of rank  $r$  on  $C$ . Put  $\mathcal{D}_E = \mathcal{D}iff(E, E) = \bigcup_i \mathcal{D}_i$ ,  $\mathcal{D}_i$  is the sheaf of differential operators of degree  $\leq i$  on  $C$ . We have  $\mathcal{D}_i/\mathcal{D}_{i-1} = \mathcal{E}nd E \otimes S^i(\Theta_C)$  where  $S^i(\Theta_C)$  is the  $i$ -th symmetric product of  $\Theta_C$ .

**Definition 3.1.** We define the *Atiyah algebra* of  $E$  as

$$\mathcal{A}_E = \{\partial \in \mathcal{D}_1 \mid \text{symb}_1(\partial) \in \text{id}_E \otimes \Theta_C \subset \mathcal{E}nd(E) \otimes \Theta_C\}.$$

Here, for  $v \in \mathcal{D}_1$ ,  $\text{symb}_1(v)$  is the symbol of the differential operator  $v$ .

We have the inclusion  $\mathcal{D}_0 = \mathcal{E}nd E \subset \mathcal{A}_E \subset \mathcal{D}_1$  and the short exact sequence

$$(8) \quad 0 \longrightarrow \mathcal{E}nd(E) \longrightarrow \mathcal{A}_E \xrightarrow{\text{symb}_1} \Theta_C \longrightarrow 0.$$

By this exact sequence, we have the following exact sequence

$$0 \longrightarrow H^1(C, \mathcal{E}nd(E)) \longrightarrow H^1(C, \mathcal{A}_E) \xrightarrow{\text{symb}_1} H^1(C, \Theta_C) \longrightarrow 0.$$

Then  $H^1(C, \mathcal{A}_E)$  implies the set of infinitesimal deformations of the pair  $(C, E)$ . Fix a positive integer  $n$ . Let  $D = t_1 + \dots + t_n$  be an effective divisor of  $C$  where  $t_1, \dots, t_n$  are distinct points of  $C$ . We put  $\mathcal{A}_E(D) := \text{symb}_1^{-1}(\Theta_C(-D))$ . Then we have the following exact sequence

$$(9) \quad 0 \longrightarrow \mathcal{E}nd(E) \longrightarrow \mathcal{A}_E(D) \xrightarrow{\text{symb}_1} \Theta_C(-D) \longrightarrow 0.$$

For a connection  $\nabla: E \rightarrow E \otimes \Omega_C^1(D)$ , we define the splitting

$$(10) \quad \iota(\nabla): \Theta_C(-D) \longrightarrow \mathcal{A}_E(D)$$

as follows. Let  $U$  be an affine open subset of  $C$  where we have a trivialization  $E|_U \cong \mathcal{O}_U^{\oplus r}$ . We denote by  $Af^{-1}df$  a connection matrix of  $\nabla$  on  $U$  where  $f$  is a local defining equation of  $t_i$  and  $A \in M_r(\mathcal{O}_U)$ . For an element  $g \frac{\partial}{\partial f} \in \Theta_C(-D)(U)$ , we define the element  $\iota(\nabla)(g \frac{\partial}{\partial f}) := g \left( \frac{\partial}{\partial f} + Af^{-1} \right) \in \mathcal{A}_E(D)(U)$ , which gives a map  $\iota(\nabla)(U): \Theta_C(-D)(U) \rightarrow \mathcal{A}_E(D)(U)$ . By this map, we obtain the splitting (10).

**3.2. Description of the relative tangent sheaf  $\Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)/N_r^{(n)}(d)}$ .** We discuss a description of the relative tangent sheaf  $\Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)/N_r^{(n)}(d)}$  by the hypercohomology of a certain complex. Let  $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$  be a universal family on  $\mathcal{C} \times_T M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$ . First, we define a complex  $\mathcal{G}^{\bullet}$  as follows:

$$\mathcal{G}^0 := \left\{ s \in \mathcal{A}_{\tilde{E}}(D(\tilde{\mathbf{t}})) \mid (s - \iota(\tilde{\nabla}) \circ \text{symb}_1(s))|_{\tilde{l}_i \times M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)}(\tilde{l}_j^{(i)}) \subset \tilde{l}_j^{(i)} \text{ for any } i, j \right\}$$

$$\mathcal{G}^1 := \left\{ s \in \mathcal{E}nd(\tilde{E}) \otimes \tilde{\Omega}^1 \mid \begin{array}{l} s(F_j^{(i)}(\tilde{E})) \subset F_j^{(i)}(\tilde{E}) \otimes \tilde{\Omega}^1 \text{ for any } i, j \text{ and the image of} \\ F_j^{(i)}(\tilde{E}) \xrightarrow{s} \tilde{E} \otimes \tilde{\Omega}^1 \xrightarrow{\text{res}_{\tilde{l}_i}} \tilde{E}_{\tilde{l}_i \times \pi^{-1}(U)} \text{ is contained in } \tilde{l}_{j+1}^{(i)} \text{ for any } i, j \end{array} \right\}$$

where

$$\tilde{\Omega}^1 := \Omega_{\mathcal{C} \times_T M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)/M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)}(D(\tilde{\mathbf{t}})) \oplus \mathcal{O}_{\mathcal{C} \times_T M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d, \nu)} d\epsilon;$$

$$F_j^{(i)}(\tilde{E}) := \ker \left( \tilde{E} \rightarrow \tilde{E}|_{\tilde{l}_i \times M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)} / \tilde{l}_j^{(i)} \right);$$

and  $\mathcal{A}_{\tilde{E}}(D(\tilde{\mathbf{t}}))$  is the relative Atiyah algebra which is the extension

$$0 \longrightarrow \mathcal{E}nd(\tilde{E}) \longrightarrow \mathcal{A}_{\tilde{E}}(D(\tilde{\mathbf{t}})) \longrightarrow \Theta_{\mathcal{C} \times_T M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)/M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)}(-D(\tilde{\mathbf{t}})) \longrightarrow 0.$$

We define a complex  $\mathcal{G}^{\bullet}$  as

$$(11) \quad \mathcal{G}^{\bullet}: \mathcal{G}^0 \xrightarrow{d^0} \mathcal{G}^1;$$

$$d^0(s) = \tilde{\nabla} \circ (s - \iota(\tilde{\nabla}) \circ \text{symb}_1(s)) - (s - \iota(\tilde{\nabla}) \circ \text{symb}_1(s)) \circ \tilde{\nabla} + (s - \iota(\tilde{\nabla}) \circ \text{symb}_1(s))d\epsilon$$

where  $\iota(\tilde{\nabla})$  is the splitting (10) associated to  $\tilde{\nabla}$ .

**Proposition 3.2.** *There exists a canonical isomorphism*

$$(12) \quad \varsigma: \Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)/N_r^{(n)}(d)} \xrightarrow{\sim} \mathbf{R}^1(\pi_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)})_*(\mathcal{G}^{\bullet}),$$

where  $\pi_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)}: \mathcal{C}_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)} \rightarrow M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$  is the natural morphism.

We show this proposition. We take an affine open set  $M \subset M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$ . We also denote by  $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$  the family on  $\mathcal{C}_M = \mathcal{C} \times_T M$  induced by the universal family. We take an affine open covering  $\mathcal{C}_M = \bigcup_{\alpha} U_{\alpha}$  such that  $\phi_{\alpha}: \tilde{E}|_{U_{\alpha}} \xrightarrow{\sim} \mathcal{O}_{U_{\alpha}}^{\oplus r}$  for any  $\alpha$ ,  $\#\{i \mid \tilde{t}_i|_{\mathcal{C}_M} \cap U_{\alpha} \neq \emptyset\} \leq 1$  for any  $\alpha$  and  $\#\{\alpha \mid \tilde{t}_i|_{\mathcal{C}_M} \cap U_{\alpha} \neq \emptyset\} \leq 1$  for any  $i$ . Take a relative tangent vector field  $v \in \Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)/N_r^{(n)}(d)}(M)$ . The field  $v$  corresponds to a member  $(C_{\epsilon}, E_{\epsilon}, \nabla_{\epsilon}, \{(l_{\epsilon})_j^{(i)}\}) \in M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)(\text{Spec } \mathcal{O}_M[\epsilon])$  such that  $(C_{\epsilon}, E_{\epsilon}, \nabla_{\epsilon}, \{(l_{\epsilon})_j^{(i)}\}) \otimes \mathcal{O}_M[\epsilon]/(\epsilon) \cong (\mathcal{C}_M, \tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ , where  $\mathcal{O}_M[\epsilon] = \mathcal{O}_M[t]/(t^2)$ . Here,  $C_{\epsilon} \rightarrow \text{Spec } \mathcal{O}_M[\epsilon]$  is the base-change of  $\mathcal{C}_M \rightarrow M$ ,  $E_{\epsilon}$  is a vector bundle on  $C_{\epsilon}$ , and

$$\nabla_{\epsilon}: E_{\epsilon} \longrightarrow E_{\epsilon} \otimes \Omega_{C_{\epsilon}/M}^1(\log(D(\tilde{\mathbf{t}})_{\mathcal{O}_M[\epsilon]}))$$

where we define the sheaf  $\Omega_{C_{\epsilon}/M}^1(\log(D(\tilde{\mathbf{t}})_{\mathcal{O}_M[\epsilon]}))$  as the coherent subsheaf of  $\Omega_{C_{\epsilon}/M}^1(D(\tilde{\mathbf{t}})_{\mathcal{O}_M[\epsilon]})$  locally generated by  $f^{-1}df$  and  $d\epsilon$  for a local defining equation  $f$  of  $D(\tilde{\mathbf{t}})_{\mathcal{O}_M[\epsilon]}$ , which is the pull-back of  $D(\tilde{\mathbf{t}})$  by the morphism  $C_{\epsilon} \rightarrow \mathcal{C}_M \rightarrow \mathcal{C}$ . Set  $U_{\alpha}^{\epsilon} := U_{\alpha} \times \text{Spec } \mathcal{O}_M[\epsilon]$ . Let

$$\mu_{\alpha\beta}(\epsilon): U_{\alpha\beta} \times \text{Spec } \mathcal{O}_M[\epsilon] \xrightarrow{\sim} U_{\alpha\beta} \times \text{Spec } \mathcal{O}_M[\epsilon]$$

be an isomorphism associated to the first-order deformation  $C_{\epsilon}$  of  $\mathcal{C}_M$ . The isomorphism  $\mu_{\alpha\beta}(\epsilon)$  satisfies

$$\mu_{\alpha\beta}(\epsilon)^*(\epsilon) = \epsilon, \quad \mu_{\alpha\beta}(\epsilon)^*(f) = f + \epsilon d_{\alpha\beta} f, \text{ for } f \in \mathcal{O}_{U_{\alpha\beta}},$$

for some  $d_{\alpha\beta} \in \Theta_{\mathcal{C}_M}(-D)(U_{\alpha\beta})$ . We describe  $d_{\alpha\beta}$  as  $d_{\alpha\beta} = \frac{\partial \mu_{\alpha\beta}(\epsilon)}{\partial \epsilon} \frac{\partial}{\partial f_{\alpha}} \in \Theta_C(-D)(U_{\alpha\beta})$ . Here,  $f_{\alpha}$  is a local defining equation of  $\tilde{t}_i|_{\mathcal{C}_M} \cap U_{\alpha}$ . Set  $\phi_{\alpha}^{\epsilon}: E_{\epsilon}|_{U_{\alpha}^{\epsilon}} \cong \mathcal{O}_{U_{\alpha}^{\epsilon}}^{\oplus r}$ . There is an isomorphism

$$\varphi_{\alpha}: E_{\epsilon}|_{U_{\alpha}^{\epsilon}} \xrightarrow[\sim]{\phi_{\alpha}^{\epsilon}} \mathcal{O}_{U_{\alpha}^{\epsilon}}^{\oplus r} \xrightarrow[\sim]{\phi_{\alpha}^{-1}} \tilde{E}|_{U_{\alpha}} \otimes \mathcal{O}_M[\epsilon]$$

such that  $\varphi_{\alpha} \otimes \mathcal{O}_M[\epsilon]/(\epsilon): E_{\epsilon} \otimes \mathcal{O}_M[\epsilon]/(\epsilon)|_{U_{\alpha}} \xrightarrow{\sim} \tilde{E}|_{U_{\alpha}} \otimes \mathcal{O}_M[\epsilon]/(\epsilon) = \tilde{E}|_{U_{\alpha}}$  is the given isomorphism and that  $\varphi_{\alpha}|_{\tilde{t}_i \otimes \mathcal{O}_M[\epsilon]}((l_{\epsilon})_j^{(i)}) = \tilde{l}_j^{(i)}|_{U_{\alpha} \times \text{Spec } \mathcal{O}_M[\epsilon]}$  if  $\tilde{t}_i|_{\mathcal{C}_M} \cap U_{\alpha} \neq \emptyset$ . Put

$$\theta_{\alpha\beta}(\epsilon): \mathcal{O}_{U_{\alpha\beta}^{\epsilon}}^{\oplus r} \xrightarrow[\sim]{(\phi_{\beta}^{\epsilon})^{-1}|_{U_{\alpha\beta}^{\epsilon}}} E_{\epsilon}|_{U_{\alpha\beta}^{\epsilon}} \xrightarrow[\sim]{\phi_{\alpha}^{\epsilon}|_{U_{\alpha\beta}^{\epsilon}}} \mathcal{O}_{U_{\alpha\beta}^{\epsilon}}^{\oplus r},$$

which is an element of  $\mathcal{E}nd(\mathcal{O}_{U_{\alpha\beta}^{\epsilon}}^{\oplus r})(U_{\alpha\beta}^{\epsilon})$ . We denote  $\theta_{\alpha\beta}(\epsilon)$  by

$$\theta_{\alpha\beta}(\epsilon) = \tilde{\theta}_{\alpha\beta} + \epsilon \frac{\partial \theta_{\alpha\beta}(\epsilon)}{\partial \epsilon} \quad \text{where } \tilde{\theta}_{\alpha\beta}, \frac{\partial \theta_{\alpha\beta}(\epsilon)}{\partial \epsilon} \in \mathcal{E}nd(\mathcal{O}_{U_{\alpha\beta}^{\epsilon}}^{\oplus r})(U_{\alpha\beta}).$$

Set  $\eta_{\alpha\beta} := \frac{\partial \theta_{\alpha\beta}(\epsilon)}{\partial \epsilon}(\tilde{\theta}_{\alpha\beta})^{-1} \in \mathcal{E}nd(\mathcal{O}_{U_{\alpha\beta}^{\epsilon}}^{\oplus r})(U_{\alpha\beta})$ . We define elements  $u_{\alpha\beta} \in \mathcal{G}^0(U_{\alpha\beta})$  and  $v_{\alpha} \in \mathcal{G}^1(U_{\alpha})$  by

$$(13) \quad u_{\alpha\beta} := (\phi_{\alpha}|_{U_{\alpha\beta}})^{-1} \circ \epsilon(d_{\alpha\beta} + \eta_{\alpha\beta}) \circ \phi_{\alpha}|_{U_{\alpha\beta}}$$

$$(14) \quad v_{\alpha} := (\varphi_{\alpha} \otimes \text{id}) \circ \nabla_{\epsilon}|_{U_{\alpha}^{\epsilon}} \circ \varphi_{\alpha}^{-1} - \tilde{\nabla}|_{U_{\alpha}^{\epsilon}},$$

respectively.

We denote  $(\phi_{\alpha}^{\epsilon} \times \text{id}) \circ \nabla_{\epsilon} \circ (\phi_{\alpha}^{\epsilon})^{-1}$  by

$$d + \left( \tilde{A}_{\alpha} + \epsilon A'_{\alpha} \right) \frac{df_{\alpha}^{\epsilon}}{f_{\alpha}^{\epsilon}} + B_{\alpha} d\epsilon$$

where  $\tilde{A}_{\alpha}$ ,  $A'_{\alpha}$ , and  $B_{\alpha}$  are elements of  $M_r(\mathcal{O}_{U_{\alpha}^{\epsilon}})$  which are independent of  $\epsilon$ . Here,  $f_{\alpha}^{\epsilon}$  is a local defining equation of  $(\tilde{t}_i \times \text{Spec } \mathcal{O}_M[\epsilon]) \cap U_{\alpha}^{\epsilon}$ . The relationship between  $df_{\alpha}^{\epsilon}$  and  $df_{\beta}^{\epsilon}$  is the following

$$df_{\beta}^{\epsilon} = \left( \frac{df_{\beta}}{df_{\alpha}} + \epsilon \frac{\partial}{\partial f_{\alpha}} \left( \frac{\partial \mu_{\beta\alpha}(\epsilon)}{\partial \epsilon} \right) \right) df_{\alpha}^{\epsilon} + \frac{\partial \mu_{\beta\alpha}(\epsilon)}{\partial \epsilon} d\epsilon.$$



**Lemma 3.3.** *We have*

$$u_{\beta\gamma} - u_{\alpha\gamma} + u_{\alpha\beta} = 0, \quad d^0(u_{\alpha\beta}) = v_\beta - v_\alpha.$$

*Thus,  $[(\{u_{\alpha\beta}\}, \{v_\alpha\})]$  determines an element  $\varsigma_M(v) \in \mathbf{H}^1(\mathcal{G}_M^\bullet)$ .*

*Proof.* In fact, we have

$$\begin{aligned} u_{\beta\gamma} &= \phi_\beta^{-1} \circ \epsilon \left( \frac{\partial \mu_{\beta\gamma}(\epsilon)}{\partial \epsilon} \frac{\partial}{\partial f_\beta} + \eta_{\beta\gamma} \right) \circ \phi_\beta \\ &= \phi_\beta^{-1} \circ \epsilon \left( \frac{\partial \mu_{\beta\gamma}(\epsilon)}{\partial \epsilon} \frac{\partial}{\partial f_\beta} + \eta_{\beta\alpha} + \tilde{\theta}_{\beta\alpha} \eta_{\alpha\gamma} \tilde{\theta}_{\beta\alpha}^{-1} + \frac{\partial \mu_{\alpha\gamma}(\epsilon)}{\partial \epsilon} \left( \frac{\partial \theta_{\beta\alpha}(\epsilon)}{\partial f_\alpha} (\tilde{\theta}_{\beta\alpha})^{-1} \right) \right) \circ \phi_\beta \\ &= \phi_\beta^{-1} \tilde{\theta}_{\beta\alpha} \circ \epsilon \left( \frac{\partial \mu_{\alpha\gamma}(\epsilon)}{\partial \epsilon} \frac{\partial}{\partial f_\alpha} + \eta_{\alpha\gamma} \right) \circ \tilde{\theta}_{\beta\alpha}^{-1} \phi_\beta - \phi_\beta^{-1} \circ \epsilon \left( \frac{\partial \mu_{\alpha\beta}(\epsilon)}{\partial \epsilon} \frac{\partial}{\partial f_\alpha} - \eta_{\beta\alpha} \right) \circ \phi_\beta \\ &= \phi_\beta^{-1} \tilde{\theta}_{\beta\alpha} \circ \epsilon \left( \frac{\partial \mu_{\alpha\gamma}(\epsilon)}{\partial \epsilon} \frac{\partial}{\partial f_\alpha} + \eta_{\alpha\gamma} \right) \circ \tilde{\theta}_{\beta\alpha}^{-1} \phi_\beta - \phi_\beta^{-1} \tilde{\theta}_{\beta\alpha} \circ \epsilon \left( \frac{\partial \mu_{\alpha\beta}(\epsilon)}{\partial \epsilon} \frac{\partial}{\partial f_\alpha} + \eta_{\alpha\beta} \right) \circ \tilde{\theta}_{\beta\alpha}^{-1} \phi_\beta \\ &= u_{\alpha\gamma} - u_{\alpha\beta} \end{aligned}$$

and

$$\begin{aligned} d^0(u_{\alpha\beta}) &= \phi_\alpha^{-1} \circ \left( \epsilon \left( d + \tilde{A}_\alpha \frac{df_\alpha}{f_\alpha} \right) \circ \left( \eta_{\alpha\beta} - \frac{\partial \mu_{\alpha\beta}(\epsilon)}{\partial \epsilon} \frac{\tilde{A}_\alpha}{f_\alpha} \right) \right. \\ &\quad \left. - \epsilon \left( \eta_{\alpha\beta} - \frac{\partial \mu_{\alpha\beta}(\epsilon)}{\partial \epsilon} \frac{\tilde{A}_\alpha}{f_\alpha} \right) \circ \left( d + \tilde{A}_\alpha \frac{df_\alpha}{f_\alpha} \right) + \left( \eta_{\alpha\beta} - \frac{\partial \mu_{\alpha\beta}(\epsilon)}{\partial \epsilon} \frac{\tilde{A}_\alpha}{f_\alpha} \right) d\epsilon \right) \circ \phi_\alpha \\ &= \phi_\alpha^{-1} \circ \left( \epsilon \left( d(\eta_{\alpha\beta}) - d \left( \frac{\partial \mu_{\alpha\beta}(\epsilon)}{\partial \epsilon} \frac{1}{f_\alpha} \right) \tilde{A}_\alpha - \frac{\partial \mu_{\alpha\beta}(\epsilon)}{\partial \epsilon} \frac{d(\tilde{A}_\alpha)}{f_\alpha} + \left( \tilde{A}_\alpha \frac{df_\alpha}{f_\alpha} \right) \eta_{\alpha\beta} - \eta_{\alpha\beta} \left( \tilde{A}_\alpha \frac{df_\alpha}{f_\alpha} \right) \right) \right. \\ &\quad \left. + \left( \eta_{\alpha\beta} - \frac{\partial \mu_{\alpha\beta}(\epsilon)}{\partial \epsilon} \frac{\tilde{A}_\alpha}{f_\alpha} \right) d\epsilon \right) \circ \phi_\alpha \\ &= v_\beta - v_\alpha. \end{aligned}$$

□

We can check that  $v \mapsto \varsigma_M(v)$  determines an isomorphism

$$\varsigma_M : \Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/N_r^{(n)}(d)}(M) \xrightarrow{\sim} \mathbf{H}^1(\mathcal{G}_M^\bullet); \quad v \mapsto \varsigma_M(v).$$

The isomorphism  $\varsigma_M$  induces a canonical isomorphism

$$(15) \quad \varsigma : \Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/N_r^{(n)}(d)} \xrightarrow{\sim} \mathbf{R}^1(\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)}^*)^*(\mathcal{G}^\bullet).$$

Then we obtain Proposition 3.2.

**3.3. Isomonodromic deformation.** Let  $\pi : M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d) \rightarrow T$  be the natural morphism. In this section, we define an algebraic splitting

$$(16) \quad D : \pi^*(\Theta_T) \longrightarrow \Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/N_r^{(n)}(d)}$$

of the tangent map  $\Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/N_r^{(n)}(d)} \rightarrow \pi^*(\Theta_T)$ . Here an image of (16) implies an algebraic vector field determined by the isomonodromic deformation. For this purpose, we introduce the following three complexes. First, we set

$$(17) \quad \begin{aligned} \mathcal{G}^0 &:= \left\{ s \in \mathcal{E}nd(\tilde{E}) \subset \mathcal{A}_{\tilde{E}}(D(\tilde{\mathbf{t}})) \mid s|_{\tilde{t}_i \times M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)}(\tilde{l}_j^{(i)}) \subset \tilde{l}_j^{(i)} \text{ for any } i, j \right\} \\ \mathcal{G}^2 &:= \left\{ s \in \mathcal{E}nd(\tilde{E}) \otimes \tilde{\Omega}^2 \mid s(F_j^{(i)}(\tilde{E})) \subset F_{j+1}^{(i)}(\tilde{E}) \otimes \tilde{\Omega}^2 \text{ for any } i, j \right\} \end{aligned}$$

where

$$\tilde{\Omega}^2 := \Omega_{\mathcal{C} \times T}^1 \otimes_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)}(D(\tilde{\mathbf{t}})) \wedge d\epsilon, \quad F_j^{(i)}(\tilde{E}) := \ker \left( \tilde{E} \rightarrow \tilde{E}|_{\tilde{t}_i \times M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)} / \tilde{l}_j^{(i)} \right).$$



We define a complex  $'\mathcal{G}^\bullet$  as

$$\begin{aligned} '\mathcal{G}^\bullet: '\mathcal{G}^0 &\xrightarrow{d^0} \mathcal{G}^1 \xrightarrow{d^1} \mathcal{G}^2; \\ 'd^0(s) &= \tilde{\nabla} \circ s - s \circ \tilde{\nabla} + sd\epsilon; \quad d^1(\omega + ad\epsilon) = d\epsilon \wedge \omega + (\tilde{\nabla} \circ a - a \circ \tilde{\nabla}) \wedge d\epsilon, \end{aligned}$$

which is nothing but the complex considered in [4, the proof of Proposition 8.1]. Here  $\mathcal{G}^1$  is defined in 3.2. We can show the following

**Lemma 3.4.** *The complex  $'\mathcal{G}^\bullet$  is exact.*

(See [4, the proof of Proposition 8.1]).

Second, we define a complex  $\tilde{\mathcal{G}}^\bullet$  as

$$\tilde{\mathcal{G}}^\bullet: \mathcal{G}^0 \xrightarrow{d^0} \mathcal{G}^1 \xrightarrow{d^1} \mathcal{G}^2; \quad d^1(\omega + ad\epsilon) = d\epsilon \wedge \omega + (\tilde{\nabla} \circ a - a \circ \tilde{\nabla}) \wedge d\epsilon.$$

Here  $\mathcal{G}^0$  and  $d_0$  are defined in 3.2. Third, we define a complex  $''\mathcal{G}^\bullet$  as

$$''\mathcal{G}^\bullet: \Theta_{\mathcal{C} \times_T M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d) / M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)}(-D(\tilde{\mathbf{t}})) \longrightarrow 0 \longrightarrow 0.$$

Then we have the natural exact sequence

$$(18) \quad 0 \longrightarrow '\mathcal{G}^\bullet \longrightarrow \tilde{\mathcal{G}}^\bullet \longrightarrow ''\mathcal{G}^\bullet \longrightarrow 0.$$

We start construction of the algebraic splitting

$$D: \pi^*(\Theta_T) \longrightarrow \Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d) / N_r^{(n)}(d)}$$

which gives an algebraic vector field determined by the isomonodromic deformation. Take any affine open set  $U \subset T$  and a vector field  $v \in H^0(U, \Theta_T)$ . Then  $v$  corresponds to a morphism  $\iota^v: \text{Spec } \mathcal{O}_U[\epsilon] \rightarrow T$  with  $\epsilon^2 = 0$  such that the composite  $U \hookrightarrow \text{Spec } \mathcal{O}_U[\epsilon] \rightarrow T$  is just the inclusion  $U \hookrightarrow T$ . We denote the restriction of the universal family to  $\mathcal{C} \times_T \pi^{-1}(U)$  simply by  $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ . Consider the fiber product  $\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon]$  with respect to the canonical projection  $\mathcal{C} \rightarrow T$  and the composite  $\text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon] \rightarrow \text{Spec } \mathcal{O}_U[\epsilon] \xrightarrow{\iota^v} T$ . We denote the pull-back of  $D(\tilde{\mathbf{t}})$  by the morphism  $\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon] \rightarrow \mathcal{C}$  simply by  $D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]}$ .

**Definition 3.5.** We call  $(\mathcal{E}, \nabla^\mathcal{E}, \{(l_\mathcal{E})_j^{(i)}\})$  a *horizontal lift* of  $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$  to  $\mathcal{C} \times_T \pi^{-1}(U)$  if

- (1)  $\mathcal{E}$  is a vector bundle on  $\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon]$ ,
- (2)  $\mathcal{E}|_{\tilde{l}_i \times \mathcal{O}_{\pi^{-1}(U)}[\epsilon]} = (l_\mathcal{E})_0^{(i)} \supset \cdots \supset (l_\mathcal{E})_r^{(i)} = 0$  is a filtration by subbundles for  $i = 1, \dots, n$ ,
- (3)  $\nabla^\mathcal{E}: \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_{\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon] / \pi^{-1}(U)}^1 \left( \log \left( D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]} \right) \right)$  is a connection satisfying
  - (a)  $\nabla^\mathcal{E}(F_j^{(i)}(\mathcal{E})) \subset F_j^{(i)}(\mathcal{E}) \otimes \Omega_{\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon] / \pi^{-1}(U)}^1 \left( \log \left( D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]} \right) \right)$ , where  $F_j^{(i)}(\mathcal{E})$  is given by  $F_j^{(i)}(\mathcal{E}) := \text{Ker} \left( \mathcal{E} \rightarrow \mathcal{E}|_{\tilde{l}_i \times \mathcal{O}_{\pi^{-1}(U)}[\epsilon] / \pi^{-1}(U)} / (l_\mathcal{E})_{j+1}^{(i)} \right)$ ,
  - (b) the curvature  $\nabla^\mathcal{E} \circ \nabla^\mathcal{E}: \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_{\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon] / \pi^{-1}(U)}^2 \left( \log \left( D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]} \right) \right)$  is zero,
  - (c)  $(\text{res}_{\tilde{l}_i \times \mathcal{O}_{\pi^{-1}(U)}[\epsilon]}(\tilde{\nabla}^\mathcal{E}) - \tilde{\nu}_j^{(i)})(l_\mathcal{E})_j^{(i)} \subset (l_\mathcal{E})_j^{(i)}$  for any  $i, j$ , where  $\tilde{\nabla}^\mathcal{E}$  is the relative connection over  $\text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon]$  induced by  $\nabla^\mathcal{E}$  and
  - (d)  $(\mathcal{E}, \tilde{\nabla}^\mathcal{E}, \{(l_\mathcal{E})_j^{(i)}\}) \otimes \mathcal{O}_{\pi^{-1}(U)}[\epsilon]/(\epsilon) \cong (\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ .

Here, we define the sheaf  $\Omega_{\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon] / \pi^{-1}(U)}^1 \left( \log \left( D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]} \right) \right)$  as the coherent subsheaf of  $\Omega_{\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon] / \pi^{-1}(U)}^1 \left( D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]} \right)$  locally generated by  $\tilde{g}^{-1}d\tilde{g}$  and  $d\epsilon$  for a local defining equation  $\tilde{g}$  of  $D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]}$  and the sheaf  $\Omega_{\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon] / \pi^{-1}(U)}^2 \left( \log \left( D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]} \right) \right)$  as the coherent subsheaf of  $\Omega_{\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon] / \pi^{-1}(U)}^2 \left( D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]} \right)$  locally generated by  $\tilde{g}^{-1}d\tilde{g} \wedge d\epsilon$ .

Let  $\mathcal{C} \times_T \pi^{-1}(U) = \bigcup_{\alpha} U_{\alpha}$  be an affine open covering such that  $\tilde{E}|_{U_{\alpha}} \xrightarrow{\sim} \mathcal{O}_{U_{\alpha}}^{\oplus r}$ , for any  $\alpha$ ,  $\#\{i \mid \tilde{t}_i|_{\mathcal{C} \times_T \pi^{-1}(U)} \cap U_{\alpha} \neq \emptyset\} \leq 1$  for any  $\alpha$  and  $\#\{\alpha \mid \tilde{t}_i|_{\mathcal{C} \times_T \pi^{-1}(U)} \cap U_{\alpha} \neq \emptyset\} \leq 1$  for any  $i$ . Assume that the parabolic connection  $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$  is locally given in the affine subset  $U_{\alpha}$  by a connection matrix  $A_{\alpha} f_{\alpha}^{-1} df_{\alpha}$ , where  $f_{\alpha}$  is a local defining equation of  $(\tilde{t}_i \times \pi^{-1}(U)) \cap U_{\alpha}$ ,  $A_{\alpha} \in M_r(\mathcal{O}_{U_{\alpha}})$ ,  $A_{\alpha}((\tilde{t}_i \times \pi^{-1}(U)) \cap U_{\alpha})$  is an upper triangular matrix and the parabolic structure  $\{\tilde{l}_j^{(i)}\}_W$  is given by  $(\tilde{l}_j^{(i)})_W = (*, *, \dots, *, 0, \dots, 0)$ . Put  $U_{\alpha}^{\epsilon} := U_{\alpha} \times \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon]$ . Let  $f_{\alpha}^{\epsilon}$  be a lift of  $f_{\alpha}$  which is a local defining equation of  $(\tilde{t}_i \times \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon]) \cap U_{\alpha}^{\epsilon}$ . We can take a vector bundle  $E_{\alpha}$  on  $U_{\alpha}^{\epsilon}$  such that there exists an isomorphism  $E_{\alpha} \otimes \mathcal{O}_{\pi^{-1}(U)}[\epsilon]/(\epsilon) \xrightarrow{\sim} \tilde{E}|_{U_{\alpha}}$ . If we denote the composite

$$\mathcal{O}_{\tilde{W}} \xrightarrow{d} \Omega_{\tilde{W}/\pi^{-1}(U)}^1 = \mathcal{O}_{\tilde{W}} d\tilde{f} \oplus \mathcal{O}_{\tilde{W}} d\epsilon \longrightarrow \mathcal{O}_{\tilde{W}} d\epsilon$$

by  $d_{\epsilon}$ , we can take a lift  $A_{\alpha}^{\epsilon} \in M_r(\mathcal{O}_{U_{\alpha}^{\epsilon}})$  of  $A_{\alpha}$  such that  $d_{\epsilon}(A_{\alpha}^{\epsilon}) = 0$ , which implies integrable, and  $A_{\alpha}^{\epsilon}((\tilde{t}_i \times \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon]) \cap U_{\alpha}^{\epsilon})$  is an upper triangular matrix. Then the connection matrix  $A_{\alpha}^{\epsilon}(f_{\alpha}^{\epsilon})^{-1} df_{\alpha}^{\epsilon}$  gives a *local* horizontal lift of  $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})|_{U_{\alpha}}$ .

**Lemma 3.6.** *An obstruction class for the existence of a global horizontal lift of  $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$  is in  $\mathbf{H}^2(\tilde{\mathcal{G}}_{\pi^{-1}(U)}^{\bullet})$ , and  $\mathbf{H}^2(\tilde{\mathcal{G}}_{\pi^{-1}(U)}^{\bullet}) = 0$ . Therefore, there exists a global horizontal lift of  $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ .*

*Proof.* For  $v \in H^0(U, \Theta_T)$ , we have an automorphism  $\mu_{\alpha\beta}(\epsilon): U_{\alpha\beta}^{\epsilon} \longrightarrow U_{\alpha\beta}^{\epsilon}$ . We take a lift  $\theta_{\beta\alpha}(\epsilon): E_{\alpha}|_{U_{\alpha\beta}^{\epsilon}} \xrightarrow{\sim} E_{\beta}|_{U_{\alpha\beta}^{\epsilon}}$  of the canonical isomorphism

$$\tilde{\theta}_{\beta\alpha}: E_{\alpha}|_{U_{\alpha\beta}^{\epsilon}} \otimes \mathcal{O}_{\pi^{-1}(U)}[\epsilon]/(\epsilon) \xrightarrow[\sim]{\phi_{\alpha}^{-1}} \tilde{E}|_{U_{\alpha\beta}} \xrightarrow[\sim]{\phi_{\beta}^{-1}} E_{\beta}|_{U_{\alpha\beta}^{\epsilon}} \otimes \mathcal{O}_{\pi^{-1}(U)}[\epsilon]/(\epsilon).$$

We put

$$\begin{aligned} u_{\alpha\beta\gamma} &:= \phi_{\alpha}|_{U_{\alpha\beta\gamma}} \circ \left( \theta_{\gamma\alpha}^{-1}(\epsilon)|_{U_{\alpha\beta\gamma}^{\epsilon}} \circ \theta_{\gamma\beta}(\epsilon)|_{U_{\alpha\beta\gamma}^{\epsilon}} \circ \theta_{\beta\alpha}(\epsilon)|_{U_{\alpha\beta\gamma}^{\epsilon}} - \text{id}_{E_{\alpha}|_{U_{\alpha\beta\gamma}^{\epsilon}}} \right) \circ \phi_{\alpha}^{-1}|_{U_{\alpha\beta\gamma}} \\ v_{\alpha\beta} &:= \phi_{\alpha}|_{U_{\alpha\beta}} \circ \left( \nabla_{\alpha}|_{U_{\alpha\beta}^{\epsilon}} - \theta_{\beta\alpha}^{-1} \circ \nabla_{\beta} \circ \theta_{\beta\alpha} \right) \circ \phi_{\alpha}^{-1}|_{U_{\alpha\beta}} \\ w_{\alpha} &:= \phi_{\alpha}|_{U_{\alpha}} \circ \left( \nabla_{\alpha}|_{U_{\alpha}^{\epsilon}} \circ \nabla_{\alpha}|_{U_{\alpha}^{\epsilon}} \right) \circ \phi_{\alpha}^{-1}|_{U_{\alpha}} = 0 \end{aligned}$$

where  $\nabla_{\alpha}$  and  $\nabla_{\beta}$  are the connection of the local horizontal lifts on  $U_{\alpha}^{\epsilon}$  and  $U_{\beta}^{\epsilon}$  constructed above. We describe  $\theta_{\gamma\alpha}^{-1}(\epsilon)$  and  $\theta_{\gamma\beta}(\epsilon)$  as follows:

$$(19) \quad \tilde{\theta}_{\gamma\alpha}^{-1} + \epsilon \left( \frac{\partial \mu_{\gamma\alpha}(\epsilon)}{\partial \epsilon} \frac{\partial \tilde{\theta}_{\alpha\gamma}}{\partial f_{\gamma}} - \tilde{\theta}_{\gamma\alpha}^{-1} \frac{\partial \theta_{\gamma\alpha}}{\partial \epsilon} \tilde{\theta}_{\gamma\alpha}^{-1} \right) \quad \text{and} \quad \tilde{\theta}_{\gamma\beta} + \epsilon \left( \frac{\partial \mu_{\beta\alpha}(\epsilon)}{\partial \epsilon} \frac{\partial \tilde{\theta}_{\gamma\beta}}{\partial f_{\beta}} + \frac{\partial \theta_{\gamma\beta}}{\partial \epsilon} \right).$$

We consider the following element of  $\mathcal{G}^0(U_{\alpha\beta\gamma})$  instead of  $u_{\alpha\beta\gamma}$ :

$$(20) \quad 'u_{\alpha\beta\gamma} := \phi_{\alpha}|_{U_{\alpha\beta\gamma}} \circ \left( ' \theta_{\gamma\alpha}^{\alpha}(\epsilon)^{-1} \circ ' \theta_{\gamma\beta}^{\alpha}(\epsilon) \circ \theta_{\beta\alpha}(\epsilon) - \text{id}_{E_{\alpha}|_{U_{\alpha\beta\gamma}^{\epsilon}}} \right) \circ \phi_{\alpha}^{-1}|_{U_{\alpha\beta\gamma}}$$

where we put

$$\begin{aligned} ' \theta_{\gamma\alpha}^{\alpha}(\epsilon)^{-1} &:= \tilde{\theta}_{\gamma\alpha}^{-1} \circ \left( \text{id} - \epsilon \left( \frac{\partial \mu_{\gamma\alpha}(\epsilon)}{\partial \epsilon} \frac{\partial}{\partial x_{\gamma}} + \frac{\partial \theta_{\gamma\alpha}}{\partial \epsilon} \tilde{\theta}_{\gamma\alpha}^{-1} \right) \right) \\ ' \theta_{\gamma\beta}^{\alpha}(\epsilon) &:= \left( \text{id} + \epsilon \left( \frac{\partial \mu_{\beta\alpha}(\epsilon)}{\partial \epsilon} \frac{\partial}{\partial x_{\beta}} + \frac{\partial \theta_{\gamma\beta}}{\partial \epsilon} \tilde{\theta}_{\gamma\beta}^{-1} \right) \right) \circ \tilde{\theta}_{\gamma\beta}. \end{aligned}$$

We can describe  $v_{\alpha\beta}$  as follows:

$$\begin{aligned} v_{\alpha\beta} &= \phi_{\alpha}|_{U_{\alpha\beta}} \circ \left( \epsilon \left( \frac{\partial \theta_{\alpha\beta}(\epsilon)}{\partial \epsilon} \tilde{A}_{\beta} \frac{df_{\beta}}{f_{\beta}} \tilde{\theta}_{\beta\alpha} + \tilde{\theta}_{\beta\alpha}^{-1} \tilde{A}_{\beta} \frac{df_{\beta}}{f_{\beta}} \frac{\partial \theta_{\beta\alpha}(\epsilon)}{\partial \epsilon} + \frac{\partial \theta_{\alpha\beta}(\epsilon)}{\partial \epsilon} (d\tilde{\theta}_{\beta\alpha}) + \tilde{\theta}_{\alpha\beta} \left( d \frac{\partial \theta_{\beta\alpha}(\epsilon)}{\partial \epsilon} \right) \right. \right. \\ &\quad \left. \left. + \tilde{\theta}_{\beta\alpha}^{-1} \left( \frac{\partial \mu_{\beta\alpha}(\epsilon)}{\partial \epsilon} \frac{\partial \tilde{A}_{\beta} f_{\beta}^{-1}}{\partial f_{\beta}} \frac{df_{\beta}}{df_{\alpha}} df_{\alpha} + \tilde{A}_{\beta} f_{\beta}^{-1} \frac{\partial^2 \mu_{\beta\alpha}(\epsilon)}{\partial f_{\alpha} \partial \epsilon} df_{\alpha} \right) \tilde{\theta}_{\beta\alpha} \right) \right. \\ &\quad \left. + \left( \tilde{\theta}_{\beta\alpha}^{-1} \frac{\partial \theta_{\beta\alpha}(\epsilon)}{\partial \epsilon} + \tilde{\theta}_{\beta\alpha}^{-1} \left( \frac{\partial \mu_{\beta\alpha}(\epsilon)}{\partial \epsilon} \tilde{A}_{\beta} f_{\beta}^{-1} \right) \tilde{\theta}_{\beta\alpha} \right) d\epsilon \right) \circ \phi_{\alpha}^{-1}|_{U_{\alpha\beta}} \end{aligned}$$

by the integrability condition  $d_\epsilon(A_\alpha^\epsilon) = 0$ . Then we can show that

$$d^0(\{u_{\alpha\beta\gamma}\}) = -d\{v_{\alpha\beta}\} \text{ and } d^1(\{v_{\alpha\beta}\}) = 0.$$

Therefore,  $[(\{u_{\alpha\beta\gamma}\}, \{v_{\alpha\beta}\}, \{w_\alpha\})]$  is an element of  $\mathbf{H}^2(\tilde{\mathcal{G}}_{\pi^{-1}(U)}^\bullet)$ , and this element is an obstruction class for the existence of a global horizontal lift of  $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ .

By Lemma 3.4 and the short exact sequence (18), we have  $\mathbf{H}^2(\tilde{\mathcal{G}}_{\pi^{-1}(U)}^\bullet) = 0$ , that is, obstruction classes for the existence of global horizontal lifts vanish.  $\square$

Next, we show that the uniqueness of the horizontal lift. Since  $\mathbf{H}^1(\mathcal{G}_{\pi^{-1}(U)}^\bullet) = \mathbf{H}^2(\mathcal{G}_{\pi^{-1}(U)}^\bullet) = 0$ , we have  $\iota: \mathbf{H}^1(\tilde{\mathcal{G}}_{\pi^{-1}(U)}^\bullet) \cong \mathbf{H}^1(\mathcal{G}_{\pi^{-1}(U)}^\bullet)$ . Let

$$\mu: \pi^*\Theta_T \longrightarrow \mathbf{R}^1(\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)}^*)^*(\mathcal{G}^\bullet)$$

be the Kodaira–Spencer map where  $\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)}: \mathcal{C}_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)} \rightarrow M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$  is the natural morphism. Let  $M$  be an affine open set of  $\pi^{-1}(U) \subset M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ . The set of horizontal lifts associated to  $v \in (\pi^*\Theta_T)(M)$  is  $\iota^{-1}(\mu(v)) \subset \mathbf{H}^1(\tilde{\mathcal{G}}_M^\bullet)$ . Then we obtain the uniqueness of the horizontal lift. We define an algebraic splitting by

$$\begin{aligned} D_M: \pi^*(\Theta_T)(M) &\longrightarrow \Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/N_r^{(n)}(d)}(M) \cong \mathbf{H}^1(\mathcal{G}_M^\bullet) \\ v &\longmapsto p(\iota^{-1}(\mu(v))) \end{aligned}$$

where  $p: \mathbf{H}^1(\tilde{\mathcal{G}}_M^\bullet) \rightarrow \mathbf{H}^1(\mathcal{G}_M^\bullet)$  is the natural  $\mathcal{O}_M$ -morphism. The algebraic splitting  $D_M$  induces the desired algebraic splitting

$$D: \pi^*(\Theta_T) \longrightarrow \Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/N_r^{(n)}(d)}.$$

The images of  $D$  imply algebraic vector fields determined by the isomonodromic deformation. We can describe images of the algebraic splitting  $D$  as follows.

**Proposition 3.7.** *Let  $[(\partial\mu_{\alpha\beta}(\epsilon)/\partial\epsilon)(\partial/\partial f_\alpha)]$  be the image of the Kodaira–Spencer map  $\mu_M$  of  $v \in (\pi^*\Theta_T)(M)$ . Put*

(21)

$$u_{\alpha\beta}^{\text{IMD}} := (\phi_\alpha|_{U_{\alpha\beta}})^{-1} \circ \left( \iota(\tilde{\nabla}) \left( \frac{\partial\mu_{\alpha\beta}(\epsilon)}{\partial\epsilon} \frac{\partial}{\partial f_\alpha} \right) \right) \circ \phi_\alpha|_{U_{\alpha\beta}} = (\phi_\alpha|_{U_{\alpha\beta}})^{-1} \circ \frac{\partial\mu_{\alpha\beta}(\epsilon)}{\partial\epsilon} \left( \frac{\partial}{\partial f_\alpha} + \tilde{A}_\alpha \tilde{f}_\alpha^{-1} \right) \circ \phi_\alpha|_{U_{\alpha\beta}}$$

(22)

$$v_\alpha^{\text{IMD}} := 0$$

where  $\iota(\tilde{\nabla})$  is the splitting (10) associated to the universal family  $\tilde{\nabla}$ . Here  $\tilde{A}_\alpha \tilde{f}_\alpha^{-1}$  is a connection matrix of  $\tilde{\nabla}$  associated to the trivialization  $\phi_\alpha$ . Then

$$D_M(v) = [(\{u_{\alpha\beta}^{\text{IMD}}\}, \{v_\alpha^{\text{IMD}}\})] \in \mathbf{H}^1(\mathcal{G}_M^\bullet) \cong \Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/N_r^{(n)}(d)}(M),$$

which is the algebraic vector field determined by the isomonodromic deformation associated to  $v$ .

*Proof.* We can show that  $\iota^{-1}(\mu_M(v)) = \{[(\{u_{\alpha\beta}^{\text{IMD}}\}, \{v_\alpha^{\text{IMD}}\}, \{0\})]\}$ . The image of  $[(\{u_{\alpha\beta}^{\text{IMD}}\}, \{v_\alpha^{\text{IMD}}\}, \{0\})]$  under the natural  $\mathcal{O}_M$ -morphism  $\mathbf{H}^1(\tilde{\mathcal{G}}_M^\bullet) \rightarrow \mathbf{H}^1(\mathcal{G}_M^\bullet)$  is  $[(\{u_{\alpha\beta}^{\text{IMD}}\}, \{v_\alpha^{\text{IMD}}\})]$ .  $\square$

**Remark 3.8.** The connection  $\nabla^\mathcal{E}$  on  $\mathcal{E}$  on  $\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon]$  satisfies the *integrability condition* (Definition 3.5 (3) (b)). The integrability implies that the relative connection associated to  $\nabla^\mathcal{E}$  is an *isomonodromic family*. (See for example [14, 0.16.6]).

#### 4. HAMILTONIAN DESCRIPTION

In this section, we give a Hamiltonian description of a vector field determined by the isomonodromic deformation. In 4.1, we construct an initial connection  $\nabla_0$ . First, for (an elementary transformation of) an underlying vector bundle of a member of  $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ , we give an injective morphism (of locally free sheaves) from some fixed vector bundle having same rank by the boundedness [5, Proposition 5.1]. (This construction is not canonical). Such a vector bundle with an injective morphism is treated in [3, Section 2]. Second, we construct a connection on the fixed vector bundle. By the injective morphism and the connection on the fixed vector bundle, we have a connection on the underlying vector bundle of a member of  $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ , which is an initial connection  $\nabla_0$ . In 4.2, we define a vector field on each affine open subset of  $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$  associated to an initial connection  $\nabla_0$ . For construction of the vector fields, since an initial connection  $\nabla_0$  has poles except for  $D(\tilde{\mathbf{t}})$ , we need some condition of a deformation of an  $n$ -pointed curve on neighborhoods of the poles except for  $D(\tilde{\mathbf{t}})$ . These vector fields are considered as vector fields associated to time variables. In 4.3, we describe the main theorem. First, we give a two form  $\omega$  on  $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$  such that the kernel  $\text{Ker}(\omega)$  implies a vector field determined by the isomonodromic deformation and  $\omega$  is the symplectic form fiberwise. Second, we define Hamiltonian functions on each affine open set of  $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ . Finally, if we take good coordinates on each affine open set of  $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ , we obtain a Hamiltonian description of a vector field determined by the isomonodromic deformation on each affine open set.

**4.1. Construction of initial connections  $\tilde{\nabla}_0^{\sigma_M}$ .** Let  $T$  be a connected noetherian scheme and  $\pi_T: \mathcal{C} \rightarrow T$  be a smooth projective morphism whose geometric fibers are curves of genus  $g$ . We take a  $T$ -ample line bundle  $\mathcal{O}_{\mathcal{C}}(1)$  on  $\mathcal{C}$ . Set  $d_{\mathcal{C}} = \deg \mathcal{O}_{\mathcal{C}_s}(1)$  for  $s \in T$ . Let  $m$  be an integer enough large and  $d'$  be an integer where  $r$  and  $d'$  are coprime. We take an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{C}}(-m) \longrightarrow \mathcal{O}_{\mathcal{C}} \longrightarrow \mathcal{O}_{\mathcal{C}}/\mathcal{O}_{\mathcal{C}}(-m) \longrightarrow 0.$$

Let  $\tilde{E}$  be the underlying vector bundle of the universal family of  $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ .

Let  $\mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)$  be a moduli functor defined in 2.4. Applying a certain elementary transformation, we obtain an isomorphism

$$(23) \quad \tau: \mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \xrightarrow{\sim} \mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d')$$

of functors, where  $d'$  and  $r$  are coprime. Then  $\tau(\mathcal{M}_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d))$  can be considered as the moduli scheme of parabolic connections of rank  $r$  and of degree  $d'$  satisfying a certain stability condition. Note that we have a commutative diagram

$$\begin{array}{ccc} M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d) & \xrightarrow{\sim} & \tau(\mathcal{M}_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)) \\ & \searrow & \swarrow \\ & T & \end{array}$$

Let  $\tilde{E}^\tau$  be the elementary transform (23) of  $\tilde{E}$ . For an open set  $M$  of  $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ , we put  $\mathcal{C}_{\tau(M)} := \mathcal{C} \times_T \tau(M)$ , which is isomorphic to  $\mathcal{C}_M$ .

**Proposition 4.1.** *There exists an open covering  $\{M\}$  of  $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$  such that, for each  $M$ , we can define elements  $\sigma_M^{(1)}, \dots, \sigma_M^{(r)} \in H^0(\mathcal{C}_{\tau(M)}, \tilde{E}_{\mathcal{C}_{\tau(M)}}^\tau(m))$  where*

- (1)  $\sigma_M := (\sigma_M^{(1)}, \dots, \sigma_M^{(r)}): \mathcal{O}_{\mathcal{C}_{\tau(M)}}^{\oplus r}(-m) \rightarrow \tilde{E}_{\mathcal{C}_{\tau(M)}}^\tau$  is injective,
- (2)  $\text{Supp}(D(\sigma_M))$  is disjoint from  $\text{Supp}(D(\tilde{\mathbf{t}})) \cup \text{Supp}(D(m))$  and,
- (3) the pull-back  $D(\sigma_M)_{\mathcal{O}_{\mathcal{C}_x}}$  of  $D(\sigma_M)$  by  $\mathcal{C}_x \rightarrow \mathcal{C}_M$  consists of distinct points for any  $x \in M$ .

Here  $D(m)$  and  $D(\sigma_M)$  are the Cartier divisors of  $\mathcal{C}_{\tau(M)}$  such that for any  $x \in \tau(M)$ ,

$$D(m)_{\mathcal{O}_{\mathcal{C}_x}} = \sum_{p \in \mathcal{C}_x} \text{length} \left( \text{Coker} \left( \mathcal{O}_{\mathcal{C}_{\tau(M)}}^{\oplus r}(-m) \rightarrow \mathcal{O}_{\mathcal{C}_{\tau(M)}}^{\oplus r} \right)_p \right) [p] \text{ and}$$

$$D(\sigma_M)_{\mathcal{O}_{\mathcal{C}_x}} = \sum_{p \in \mathcal{C}_x} \text{length} \left( \text{Coker}(\sigma_M)_p \right) [p],$$

respectively. We also denote by  $D(m)$  and  $D(\sigma_M)$  the inverse images of  $D(m)$  and  $D(\sigma_M)$  under the isomorphism  $\mathcal{C}_M \xrightarrow{\sim} \mathcal{C}_{\tau(M)}$ , respectively.

For a proof of this proposition, we recall the construction of a universal family of  $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$  due to Inaba [4, Proof of Theorem 2.1]. We can take a vector space  $V$  such that there exists a surjective  $V \otimes \mathcal{O}_{\mathcal{C}_s}(-m) \rightarrow E$  such that  $V \otimes k(s) \rightarrow H^0(E(m))$  is isomorphism and  $h^i(E(m)) = 0$  for  $i > 0$  for any member  $(E, \nabla, \{l_j^{(i)}\}) \in \tau(\mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d))$  over  $s \in T$ . We put  $P(n) := \chi(E(n))$ . Let  $\mathcal{E}$  be a universal family on  $\mathcal{C} \times_T \text{Quot}_{V \otimes \mathcal{O}_{\mathcal{C}(m)}/\mathcal{C}/T \times N_r^{(n)}(d)}^P$ . Then we can construct a scheme  $R$  over  $\text{Quot}_{V \otimes \mathcal{O}_{\mathcal{C}(m)}/\mathcal{C}/T \times N_r^{(n)}(d)}^P$  which parametrize connection  $\nabla: \mathcal{E}_s \rightarrow \mathcal{E}_s \otimes \Omega_{\mathcal{C}_s}^1(D(\tilde{\mathbf{t}}_s))$  and parabolic structures  $\mathcal{E}_s|_{(\tilde{\mathbf{t}}_i)_s} = l_0^{(i)} \supset \dots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0$  such that  $(\text{res}_{(\tilde{\mathbf{t}}_i)_s}(\nabla) - (\tilde{\lambda}_j^{(i)})_s)(l_j^{(i)}) \subset l_{j+1}^{(i)}$  for any  $i, j$ . We put  $G := \text{PGL}(V)$ . The algebraic group  $G$  canonically acts on  $R$  and for some open subscheme  $R^s$  of  $R$ , there exists a canonical morphism

$$R^s \longrightarrow \tau(\mathcal{M}_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d))$$

which becomes a principal  $G$ -bundle. For a certain line bundle  $\mathcal{L}$  on  $R^s$ , the vector bundle  $\mathcal{E}_{\mathcal{C} \times_T R^s} \otimes \mathcal{L}$  on  $\mathcal{C} \times_T R^s$  descends to a vector bundle on  $\mathcal{C} \times_T \tau(\mathcal{M}_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d))$ , since  $r$  and  $d'$  are coprime.

*Proof of Proposition 4.1.* For the principal  $G$ -bundle  $R^s \rightarrow \tau(\mathcal{M}_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d))$ , we take a surjective étale morphism  $\tilde{M} \rightarrow \tau(\mathcal{M}_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d))$  and a  $G$ -equivariant isomorphism  $\varphi_{\tilde{M}}: \tilde{M} \times G \rightarrow R_{\tilde{M}}^s = \tilde{M} \times_{\tau(\mathcal{M}_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d))} R^s$ . We put  $\mathcal{O}_{\mathcal{C} \times_T R_{\tilde{M}}^s}(1) := \mathcal{O}_{\mathcal{C} \times_T R_{\tilde{M}}^s} \otimes \mathcal{O}_{\mathcal{C}}(1)$ . Let

$$\sigma_V: V \otimes \mathcal{O}_{\mathcal{C} \times_T (\tilde{M} \times G)}(-m) \longrightarrow \mathcal{E}_{\mathcal{C} \times_T (\tilde{M} \times G)}$$

be the pull-back of the universal family on  $\mathcal{C} \times_T \text{Quot}_{V \otimes \mathcal{O}_{\mathcal{C}(m)}/\mathcal{C}/T \times N_r^{(n)}(d)}^P$ . We put  $\mathcal{C}_{M' \times G} := \mathcal{C} \times_T (M' \times G)$ .

We take a pair  $(M, W_r)$  of an open set of  $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$  and an  $r$ -dimensional subspace of  $V$  as follows. We consider the immersion  $\tilde{M} \rightarrow \tilde{M} \times G; u \mapsto (u, \text{id})$ . Let

$$\sigma_V^0: V \otimes \mathcal{O}_{\mathcal{C} \times_T \tilde{M}}(-m) \longrightarrow \mathcal{E}_{\mathcal{C} \times_T \tilde{M}}$$

be the pull-back of  $\sigma_V$  by  $\mathcal{C} \times_T \tilde{M} \rightarrow \mathcal{C} \times_T (\tilde{M} \times G)$ . We take the determinant  $\det(\sigma_V^0)$ . Then we have global sections of  $\det(\mathcal{E}_{\mathcal{C} \times_T \tilde{M}}(m))$ , which give an immersion of the family of curves  $\mathcal{C} \times_T \tilde{M}$  to some relative projective space. We take a global section of  $\det(\mathcal{E}_{\mathcal{C} \times_T \tilde{M}}(m))$  and an open set  $M'$  of  $\tilde{M}$  such that, on  $\mathcal{C} \times_T M'$ , the corresponding hyperplane section satisfies that

- the support of the hyperplane section is disjoint from  $\text{Supp}(D(\tilde{\mathbf{t}})_{\mathcal{O}_{\mathcal{C} \times_T M'}}) \cup \text{Supp}(D(m)_{\mathcal{O}_{\mathcal{C} \times_T M'}})$  and
- the pull-back of the hyperplane section by  $\mathcal{C} \times_T x \rightarrow \mathcal{C} \times_T M'$  consists of distinct points for any  $x \in M'$ .

We take an  $r$ -dimensional subspace  $W_r \subset V$  associated to such a hyperplane section, and we can take an open set  $M$  of  $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$  such that the inverse image of  $M$  under the composition  $\tilde{M} \rightarrow \tau(\mathcal{M}_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)) \rightarrow M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$  is contained in  $M'$ . We take a collection of such pairs  $\{(M, W_r)\}$  such that  $\{M\}$  is an open covering of the moduli space  $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$ .

Let  $(M, W_r)$  be a pair as above. Let  $M'$  be the inverse image of the open set  $M$  under the composition  $\tilde{M} \rightarrow \tau(\mathcal{M}_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)) \rightarrow M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ . We consider the  $\mathcal{O}_{\mathcal{C}_{M' \times G}}$ -morphism

$$(24) \quad \begin{aligned} \sigma_{W_r, M'}: W_r \otimes \mathcal{O}_{\mathcal{C}_{M' \times G}}(-m) &\longrightarrow \mathcal{E}_{\mathcal{C}_{M' \times G}} \\ (f_1 \mathbf{e}_1 \otimes \mathbf{f} + \cdots + f_r \mathbf{e}_r \otimes \mathbf{f})_{(u, g)} &\longmapsto (f_1 \sigma_V(g^{-1} \cdot \mathbf{e}_1 \otimes \mathbf{f}) + \cdots + f_r \sigma_V(g^{-1} \cdot \mathbf{e}_r \otimes \mathbf{f}))_{(u, g)} \end{aligned}$$

where  $(u, g) \in M' \times G$ . Here  $\mathbf{e}_1, \dots, \mathbf{e}_r$  is a basis of  $W_r$ , and  $f_1, \dots, f_r$  are elements of  $\mathcal{O}_U$  where  $U$  is an open set of  $\mathcal{C}_{M' \times G}$  such that there is a trivialization  $\mathcal{O}_U \rightarrow \mathcal{O}_{\mathcal{C}_{M' \times G}}(-m)|_U$ ;  $f(u, g) \mapsto f(u, g)\mathbf{f}$ . Let  $\mathcal{L}'$  be the pull-back of the line bundle  $\mathcal{L}$  by  $R_{M'}^s \rightarrow R^s$ . Since  $(\sigma_{W_r, M'} \otimes \mathcal{L}')(\mathbf{e}_i)$  ( $i = 1, \dots, r$ ) descend for  $\mathcal{C} \times_T (M' \times G) \rightarrow \mathcal{C} \times_T M' \rightarrow \mathcal{C}_{\tau(M)}$  and  $\mathcal{L}'$  is locally trivial, we have  $\sigma_M^{(i)} \in H^0(\mathcal{C}_{\tau(M)}, \tilde{E}_{\mathcal{C}_{\tau(M)}}^\tau(m))$  satisfying the conditions (1), (2), and (3).  $\square$

Let  $d_m$  be the relative connection induced by  $0 \rightarrow \mathcal{O}_{\mathcal{C}_{\tau(M)}}^{\oplus r}(-m) \rightarrow \mathcal{O}_{\mathcal{C}_{\tau(M)}}^{\oplus r}$  and the relative exterior derivative  $d_{\mathcal{C}_{\tau(M)}/\tau(M)}$  on  $\mathcal{O}_{\mathcal{C}_{\tau(M)}}^{\oplus r}$ , that is, the diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{C}_{\tau(M)}}^{\oplus r}(-m) & \xrightarrow{d_m} & \mathcal{O}_{\mathcal{C}_{\tau(M)}}^{\oplus r}(-m) \otimes \Omega_{\mathcal{C}_{\tau(M)}/\tau(M)}^1(D(m)_{red}) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{C}_{\tau(M)}}^{\oplus r} & \xrightarrow{d_{\mathcal{C}_{\tau(M)}/\tau(M)}} & \mathcal{O}_{\mathcal{C}_{\tau(M)}}^{\oplus r} \otimes \Omega_{\mathcal{C}_{\tau(M)}/\tau(M)}^1(D(m)_{red}) \end{array}$$

is commutative.

**Definition 4.2.** For each open set  $M$  of Proposition 4.1, we fix  $\sigma_M^{(1)}, \dots, \sigma_M^{(r)} \in H^0(\mathcal{C}_{\tau(M)}, \tilde{E}_{\mathcal{C}_{\tau(M)}}^\tau(m))$  as in this proposition. We define a *relative initial connection*

$$(25) \quad \tilde{\nabla}_0^{\sigma_M}: \tilde{E}_{\mathcal{C}_M} \longrightarrow \tilde{E}_{\mathcal{C}_M} \otimes \Omega_{\mathcal{C}_M/M}^1(D(\tilde{\mathbf{t}}) + D(m)_{red} + D(\sigma_M))$$

by the elementary transform (23) of the relative connection induced by  $0 \rightarrow \mathcal{O}_{\mathcal{C}_{\tau(M)}}^{\oplus r}(-m) \xrightarrow{\sigma_M} \tilde{E}_{\mathcal{C}_{\tau(M)}}^\tau$  and the relative connection  $d_m$  on  $\mathcal{O}_{\mathcal{C}_{\tau(M)}}^{\oplus r}(-m)$ .

**4.2. Algebraic vector fields associated to  $\tilde{\nabla}_0^{\sigma_M}$ .** Let  $T$  be a connected noetherian scheme and  $\pi_T: \mathcal{C} \rightarrow T$  be a smooth projective morphism whose geometric fibers are curves of genus  $g$ . We take an open covering  $\{M\}$  of Proposition 4.1, and we fix  $\sigma_M^{(1)}, \dots, \sigma_M^{(r)} \in H^0(\mathcal{C}_{\tau(M)}, \tilde{E}_{\mathcal{C}_{\tau(M)}}^\tau(m))$  for each  $M$  as in this proposition. We assume that these open sets  $M$  are affine. We put  $D(\tilde{\mathbf{p}}) := D(m)_{red} + D(\sigma_M)$ . In this section we show the following

**Proposition 4.3.** *Let  $\tilde{\nabla}_0^{\sigma_M}$  be the relative initial connection of Definition 4.2. For  $\mu \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}})))$ , we take a lift  $\hat{\mu} \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}) - 2D(\tilde{\mathbf{p}})))$  as (26) below. Then for the lift  $\hat{\mu}$ , we can construct an algebraic vector field associated to  $\tilde{\nabla}_0^{\sigma_M}$  on  $M$  (Lemma 4.4 below).*

We take  $\mu = [\{(\partial\mu_{\alpha\beta}/\partial\epsilon)\partial/\partial f_\alpha\}_{\alpha,\beta}] \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}})))$  where  $D(\tilde{\mathbf{t}}) = \{(U_\alpha, f_\alpha)\}_\alpha$  is a Cartier divisor. Let  $D(m)_{red} = \{(U_\alpha, g_\alpha)\}_\alpha$  and  $D(\sigma_M) = \{(U_\alpha, h_\alpha)\}_\alpha$  be the Cartier divisors. Here let  $\{U_\alpha\}_\alpha$  be an affine open covering of  $\mathcal{C}_M$  such that

$$\phi_\alpha: \tilde{E}|_{U_\alpha} \xrightarrow{\sim} \mathcal{O}_{U_\alpha}^{\oplus r}$$

for any  $\alpha$ , and we assume that  $\#\{i \mid \tilde{t}_i|_{\mathcal{C}_M} \cap U_\alpha \neq \emptyset\} \leq 1$  for any  $\alpha$  and  $\#\{\alpha \mid \tilde{t}_i|_{\mathcal{C}_M} \cap U_\alpha \neq \emptyset\} \leq 1$  for any  $i$ . Here  $\{\tilde{t}_i\}$  is the set of the supports of the Cartier divisor  $D(\tilde{\mathbf{t}}) + D(\tilde{\mathbf{p}})$ .

Since there exists an exact sequence

$$H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}) - 2D(\tilde{\mathbf{p}}))) \longrightarrow H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}))) \longrightarrow 0,$$

we choose a lift

$$(26) \quad \hat{\mu} = \left[ \left\{ \frac{\partial \widehat{\mu_{\alpha\beta}(\epsilon)}}{\partial \epsilon} \frac{\partial}{\partial f_\alpha} \right\}_{\alpha,\beta} \right] \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}) - 2D(\tilde{\mathbf{p}})))$$

of  $\mu \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}})))$  such that for any  $x \in M$ , the image of the restriction  $(\hat{\mu})_x$  under

$$(27) \quad H^1(\mathcal{C}_x, \Theta_{\mathcal{C}_x}(-D(\tilde{\mathbf{t}}) - 2D(\tilde{\mathbf{p}}))) \cong \left( \bigoplus_{p \in \mathcal{C}_x} \text{Coker}(a_x)_p \right) / k(\mathcal{C}_x) \longrightarrow \left( \bigoplus_{p \in \text{Supp } D(\tilde{\mathbf{p}})} \text{Coker}(a_x)_p \right) / k(\mathcal{C}_x)$$

vanishes. Here the isomorphism in (27) is given by the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{C}_x}(-K_{\mathcal{C}_x} - D(\tilde{\mathbf{t}}) - 2D(\tilde{\mathbf{p}})) \xrightarrow{a_x} \mathcal{K}_{\mathcal{C}_x} \longrightarrow \text{Coker}(a_x) \longrightarrow 0$$

where  $\mathcal{K}_{\mathcal{C}_x}$  the constant sheaf whose sections are the function field  $k(\mathcal{C}_x)$ .

Now we define an algebraic vector field associated to  $\tilde{\nabla}_0^{\sigma_M}$  and  $\hat{\mu}$ . Let  $\{(U_\alpha, z_\alpha)\}_\alpha$  be the Cartier divisor  $D(\tilde{\mathbf{t}}) + D(\tilde{\mathbf{p}})$ , and let  $\tilde{A}_\alpha^0 z_\alpha^{-1} dz_\alpha$  be a connection matrix of  $\tilde{\nabla}_0^{\sigma_M}$  on  $U_\alpha$  associated to the trivialization  $\phi_\alpha$ . First, we set

$$(28) \quad u_{\alpha\beta}^{\hat{\mu}\nabla_0} := (\phi_\alpha|_{U_{\alpha\beta}})^{-1} \circ \frac{\partial \widehat{\mu_{\alpha\beta}(\epsilon)}}{\partial \epsilon} \frac{\partial z_\alpha}{\partial f_\alpha} \left( \frac{\partial}{\partial z_\alpha} + \tilde{A}_\alpha^0 z_\alpha^{-1} \right) \circ \phi_\alpha|_{U_{\alpha\beta}} \in \mathcal{G}^0(U_{\alpha\beta}).$$

Next we put

$$(29) \quad \hat{u}_{\alpha\beta}^{\text{IMD}} := (\phi_\alpha|_{U_{\alpha\beta}})^{-1} \circ \iota(\tilde{\nabla}) \left( \frac{\partial \widehat{\mu_{\alpha\beta}(\epsilon)}}{\partial \epsilon} \frac{\partial}{\partial f_\alpha} \right) \circ \phi_\alpha|_{U_{\alpha\beta}}, \quad \hat{v}_\alpha^{\text{IMD}} := 0.$$

Let  $(C_\epsilon, E_\epsilon, \nabla_\epsilon, \{(\ell_\epsilon)_j^{(i)}\}) \in M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)(\text{Spec } \mathcal{O}_M[\epsilon])$  be a first-order deformation of  $(\mathcal{C}_M, \tilde{E}, \tilde{\nabla}, \{\tilde{\ell}_j^{(i)}\})$  associated to  $[(\{\hat{u}_{\alpha\beta}^{\text{IMD}}\}, \{\hat{v}_\alpha^{\text{IMD}}\})]$ . Here we consider the element  $[(\{\hat{u}_{\alpha\beta}^{\text{IMD}}\}, \{\hat{v}_\alpha^{\text{IMD}}\})]$  as an element of  $\mathbf{H}^1(\mathcal{G}_M^\bullet)$ . We have a collection of isomorphisms  $\hat{\mu}_{\alpha\beta}(\epsilon): U_{\alpha\beta}^\epsilon \rightarrow U_{\alpha\beta}^\epsilon$  associated to  $C_\epsilon$  such that

$$(30) \quad \hat{\mu}_{\alpha\beta}(\epsilon)^*(\epsilon) = \epsilon, \quad \hat{\mu}_{\alpha\beta}(\epsilon)^*(f) = f + \epsilon \frac{\partial \widehat{\mu_{\alpha\beta}(\epsilon)}}{\partial \epsilon} \frac{\partial}{\partial f_\alpha} f$$

for  $f \in \mathcal{O}_{U_{\alpha\beta}^\epsilon}$ . Let  $D_\epsilon(\tilde{\mathbf{t}}) = \{(U_\alpha^\epsilon, f_\alpha^\epsilon)\}$ ,  $D_\epsilon(m) = \{(U_\alpha^\epsilon, g_\alpha^\epsilon)\}$ , and  $D_\epsilon(\sigma_M) = \{(U_\alpha^\epsilon, h_\alpha^\epsilon)\}$  be the pull-backs of the Cartier divisors  $D(\tilde{\mathbf{t}})$ ,  $D(m)$ , and  $D(\sigma_M)$  by the morphism  $C_\epsilon \rightarrow \mathcal{C}_M$ , respectively. Here  $g_\alpha^\epsilon$  and  $h_\alpha^\epsilon$  are pull-backs of  $g_\alpha$  and  $h_\alpha$  by the natural morphism  $U_\alpha^\epsilon \rightarrow U_\alpha$ , respectively. Let  $\phi_\alpha^\epsilon: E_\epsilon|_{U_\alpha^\epsilon} \cong \mathcal{O}_{U_\alpha^\epsilon}^{\oplus r}$  be a trivialization of  $E_\epsilon$ . We set

$$\varphi_\alpha: E_\epsilon|_{U_\alpha^\epsilon} \xrightarrow[\sim]{\phi_\alpha^\epsilon} \mathcal{O}_{U_\alpha^\epsilon}^{\oplus r} \xrightarrow[\sim]{\phi_\alpha} \tilde{E}|_{U_\alpha} \otimes \mathcal{O}_M[\epsilon].$$

We define a first-order deformation of the relative initial connection  $\tilde{\nabla}_0^{\sigma_M}$  as follows. Put  $U_{\sigma_M} := \bigcup_{\{\alpha\}} U_\alpha$  where  $\{\alpha\}$  is the set of indexes such that  $U_\alpha \cap \text{Supp}(D(\sigma_M)) \neq \emptyset$ , and put  $U_{\sigma_M}^\epsilon = U_{\sigma_M} \times \text{Spec } \mathcal{O}_M[\epsilon]$ . Since we take the lift  $\hat{\mu}$ , we can consider  $U_{\sigma_M}^\epsilon$  as an open subset of  $C_\epsilon$ . Since the lift  $\hat{\mu}$  vanishes by the map (27) and  $E_\epsilon$  is an underlying vector bundle of the first-order deformation associated to  $(\{\hat{u}_{\alpha\beta}^{\text{IMD}}\}, \{\hat{v}_\alpha^{\text{IMD}}\})$ , we can assume that

$$(31) \quad E_\epsilon|_{U_{\sigma_M}^\epsilon} = \tilde{E}|_{U_{\sigma_M}} \otimes \mathcal{O}_M[\epsilon].$$

Let  $(\tilde{E}_{\mathcal{C}_{\tau(M)}}^\tau, \{\phi_\alpha^\tau\}_\alpha)$  and  $(E_\epsilon^\tau, \{(\phi_\alpha^\epsilon)^\tau\}_\alpha)$  be the elementary transformations (23) of  $(\tilde{E}, \{\phi_\alpha\}_\alpha)$  and  $(E_\epsilon, \{\phi_\alpha^\epsilon\}_\alpha)$ , respectively. Let  $\sigma_M^{(i)}$  ( $i = 1, \dots, r$ ) be global sections of  $\tilde{E}_{\mathcal{C}_{\tau(M)}}^\tau(m)$  given in Proposition 4.1. We define a first-order deformation of  $\sigma_M^{(i)}$  by the inverse image of  $\sigma_M^{(i)}(\mathcal{C}_{\tau(M)}) \subset \tilde{E}_{\mathcal{C}_{\tau(M)}}^\tau(m)$  under the morphism  $j_\epsilon$ :

$$\begin{array}{ccc} E_\epsilon^\tau(m) & \xrightarrow{j_\epsilon} & \tilde{E}_{\mathcal{C}_{\tau(M)}}^\tau(m) \\ \downarrow & & \uparrow \sigma_M^{(i)} \\ \mathcal{C} \times_T \text{Spec } \mathcal{O}_{\tau(M)}[\epsilon] & \longrightarrow & \mathcal{C}_{\tau(M)} \end{array}$$

where  $E_\epsilon^\tau(m)$  is the tensor product of  $E_\epsilon^\tau$  and the line bundle associated to  $D_\epsilon(m)$ . We denote by  $\hat{\sigma}_M^{(i)}$  this first-order deformation of  $\sigma_M^{(i)}$ . We put  $C_\epsilon^\tau := \mathcal{C} \times_T \text{Spec } \mathcal{O}_{\tau(M)}[\epsilon]$ . These first-order deformations



$\hat{\sigma}_M^{(i)}$  ( $i = 1, \dots, r$ ) gives an injective  $\mathcal{O}_{C_\epsilon^\tau}$ -morphism

$$(32) \quad \hat{\sigma}_M^\epsilon: W_r \otimes \mathcal{O}_{C_\epsilon^\tau}(-m) \longrightarrow E_\epsilon^\tau.$$

Let

$$\nabla_{0,\epsilon}^{\sigma_M, \text{IMD}}: E_\epsilon \longrightarrow E_\epsilon \otimes \Omega_{C_\epsilon/M}^1 \left( \log \left( D_\epsilon(\tilde{\mathbf{t}}) + D_\epsilon(m)_{\text{red}} + D_\epsilon(\sigma_M) \right) \right)$$

be the first-order deformation of the relative initial connection  $\tilde{\nabla}_0^{\sigma_M}$  induced by  $\hat{\sigma}_M^\epsilon$  and the elementary transformation (23). Second, we set

$$(33) \quad v_\alpha^{\hat{\mu}\nabla_0} := (\varphi_\alpha \otimes \text{id}) \circ (\nabla_{0,\epsilon}^{\sigma_M, \text{IMD}}) \circ \varphi_\alpha^{-1} - \tilde{\nabla}_0^{\sigma_M}.$$

By  $u_{\alpha\beta}^{\hat{\mu}\nabla_0}$  and  $v_\alpha^{\hat{\mu}\nabla_0}$ , we can define an algebraic vector field:

**Lemma 4.4.** *The pair  $\left[ \left( \left\{ u_{\alpha\beta}^{\hat{\mu}\nabla_0} \right\}, \left\{ -v_\alpha^{\hat{\mu}\nabla_0} \right\} \right) \right]$  is an element of  $\mathbf{H}^1(\mathcal{G}_M^\bullet)$ , that is, this pair is an algebraic vector field on  $M$ .*

*Proof.* We show that  $\{v_\alpha^{\hat{\mu}\nabla_0}\}$  has no pole on the supports of  $D(m)$  and  $D(\sigma_M)$ . We consider the injective  $\mathcal{O}_{C_\epsilon^\tau}$ -morphism (32), denoted by  $\hat{\sigma}_M^\epsilon$ . We also describe  $U_\alpha^\epsilon = \tau(U_\alpha) \times \text{Spec } \mathcal{O}_{\tau(M)}[\epsilon]$ . We take a basis  $e_1 \otimes g_\alpha^\epsilon, \dots, e_r \otimes g_\alpha^\epsilon$  of  $W_r \otimes \mathcal{O}_{C_\epsilon^\tau}(-m)|_{U_\alpha^\epsilon}$ . Here  $\mathcal{O}_{U_\alpha^\epsilon} \rightarrow \mathcal{O}_{C_\epsilon^\tau}(-m)|_{U_\alpha^\epsilon}; f \mapsto f g_\alpha^\epsilon$  gives a trivialization of  $\mathcal{O}_{C_\epsilon^\tau}(-m)$  on  $U_\alpha^\epsilon$ . We put  $\mathbf{s}_{\alpha,i}^\epsilon := \hat{\sigma}_M^\epsilon(e_i \otimes g_\alpha^\epsilon)$ ,  $i = 1, \dots, r$ .

We consider an affine open set  $U_\alpha^\epsilon$  where  $U_\alpha^\epsilon \cap \text{Supp}(D_\epsilon(\sigma_M)) = \emptyset$ . In this case, the  $\mathcal{O}_{U_\alpha^\epsilon}$ -morphism

$$\mathcal{O}_{U_\alpha^\epsilon}^{\oplus r} \longrightarrow E_\epsilon^\tau|_{U_\alpha^\epsilon}; \quad (f_1, \dots, f_r) \longmapsto f_1 \mathbf{s}_{\alpha,1}^\epsilon + \dots + f_r \mathbf{s}_{\alpha,r}^\epsilon$$

is an isomorphism, which gives a trivialization  $\psi_\alpha^\epsilon: E_\epsilon^\tau|_{U_\alpha^\epsilon} \rightarrow \mathcal{O}_{U_\alpha^\epsilon}^{\oplus r}$ . We put  $\psi_\alpha = \psi_\alpha^\epsilon \otimes \mathcal{O}_M[\epsilon]/(\epsilon)$ . We can describe the connection  $d_m$  on  $W_r \otimes \mathcal{O}_{C_\epsilon^\tau}(-m)|_{U_\alpha^\epsilon}$  by

$$d_m|_{U_\alpha^\epsilon} = d + \text{diag}(-g_\alpha^\epsilon d(g_\alpha^\epsilon)^{-1}, \dots, -g_\alpha^\epsilon d(g_\alpha^\epsilon)^{-1}).$$

Then on  $U_\alpha^\epsilon$  where  $U_\alpha^\epsilon \cap \text{Supp}(D_\epsilon(\tilde{\mathbf{t}}) + D_\epsilon(\sigma_M)) = \emptyset$ , the relative initial connection  $\nabla_{0,\epsilon}^{\sigma_M, \text{IMD}}$  is described as

$$(34) \quad (\psi_\alpha^\epsilon \otimes \text{id}) \circ (\nabla_{0,\epsilon}^{\sigma_M, \text{IMD}}) \circ (\psi_\alpha^\epsilon)^{-1} = d_m|_{U_\alpha^\epsilon}.$$

Note that we consider the trivializations  $\psi_\alpha^\epsilon$  (resp.  $\psi_\alpha$ ) of  $E_\epsilon^\tau$  (resp.  $E_{C_\epsilon^\tau(M)}^\tau$ ) as the trivializations of  $E_\epsilon$  (resp.  $E_{C_M}$ ) on  $U_\alpha^\epsilon$ , since the elementary transformation (23) is supported on  $D(\tilde{\mathbf{t}})$ . Then  $(\psi_\alpha^{-1} \circ \psi_\alpha^\epsilon \otimes \text{id}) \circ (\nabla_{0,\epsilon}^{\sigma_M, \text{IMD}}) \circ (\psi_\alpha^{-1} \circ \psi_\alpha^\epsilon)^{-1} - \tilde{\nabla}_0^{\sigma_M}$  has no pole on the supports of  $D(m)$ .

Next we consider an affine open set  $U_\alpha^\epsilon$  where  $U_\alpha^\epsilon \cap \text{Supp}(D_\epsilon(\sigma_M)) \neq \emptyset$ . Note that

$$(\mathbf{s}_{\alpha,1}^\epsilon)_{\tilde{p}_i \times \text{Spec } \mathcal{O}_{\tau(M)}[\epsilon]}, \dots, (\mathbf{s}_{\alpha,r}^\epsilon)_{\tilde{p}_i \times \text{Spec } \mathcal{O}_{\tau(M)}[\epsilon]}$$

are linearly dependent where  $\tilde{p}_i$  is a component of the support of  $D(\sigma_M)$  on  $\tau(U_\alpha)$ . We can assume that  $(\mathbf{s}_{\alpha,2}^\epsilon)_{\tilde{p}_i \times \text{Spec } \mathcal{O}_{\tau(M)}[\epsilon]}, \dots, (\mathbf{s}_{\alpha,r}^\epsilon)_{\tilde{p}_i \times \text{Spec } \mathcal{O}_{\tau(M)}[\epsilon]}$  are linearly independent by exchange of the indexes, and

$$(\mathbf{s}_{\alpha,1}^\epsilon)_{\tilde{p}_i \times \text{Spec } \mathcal{O}_{\tau(M)}[\epsilon]} = - \sum_{k=2}^r a_{k,\alpha}^\epsilon (\mathbf{s}_{\alpha,k}^\epsilon)_{\tilde{p}_i \times \text{Spec } \mathcal{O}_{\tau(M)}[\epsilon]}.$$

By the assumption (31) and the definition of  $\hat{\sigma}_M^\epsilon$ , we have that  $a_{k,\alpha}^\epsilon$  ( $k = 2, \dots, r$ ) are elements of  $\mathcal{O}_{\tau(M)}$ , that is,  $\epsilon$ -terms of  $a_{k,\alpha}^\epsilon$  vanish. We also denote by  $a_{k,\alpha}^\epsilon$  the pull-back of  $a_{k,\alpha}^\epsilon \in \mathcal{O}_{\tau(M)}$  by  $U_\alpha^\epsilon \rightarrow \text{Spec } \mathcal{O}_{\tau(M)}[\epsilon]$ . We take  $\hat{\mathbf{s}}_{\alpha,1}^\epsilon \in E_\epsilon^\tau|_{U_\alpha^\epsilon}$  such that  $\hat{\mathbf{s}}_{\alpha,1}^\epsilon, \mathbf{s}_{\alpha,2}^\epsilon, \dots, \mathbf{s}_{\alpha,r}^\epsilon$  give a trivialization  $\psi_\alpha^\epsilon: E_\epsilon^\tau|_{U_\alpha^\epsilon} \rightarrow \mathcal{O}_{U_\alpha^\epsilon}^{\oplus r}$  of  $E_\epsilon^\tau|_{U_\alpha^\epsilon}$  and we can describe  $\mathbf{s}_{\alpha,1}^\epsilon$  as  $h_\alpha^\epsilon \hat{\mathbf{s}}_{\alpha,1}^\epsilon + a_2^\epsilon \mathbf{s}_{\alpha,2}^\epsilon + \dots + a_r^\epsilon \mathbf{s}_{\alpha,r}^\epsilon$ . We put  $\psi_\alpha = \psi_\alpha^\epsilon \otimes \mathcal{O}_M[\epsilon]/(\epsilon)$ . We set

$$(35) \quad T_\alpha^\epsilon := \begin{pmatrix} h_\alpha^\epsilon & 0 & 0 & \cdots & 0 \\ a_{2,\alpha}^\epsilon & 1 & 0 & \cdots & 0 \\ a_{3,\alpha}^\epsilon & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{r,\alpha}^\epsilon & 0 & 0 & \cdots & 1 \end{pmatrix},$$

which gives a commutative diagram

$$\begin{array}{ccc} W_r \otimes \mathcal{O}_{\mathcal{C}_\tau}(-m)|_{U_\alpha^\epsilon} & \longrightarrow & \mathcal{O}_{U_\alpha^\epsilon}^{\oplus r} \\ \downarrow \hat{\sigma}_{W_r}^\epsilon & & \downarrow T_\alpha^\epsilon \\ E_\epsilon^\tau|_{U_\alpha^\epsilon} & \xrightarrow{\psi_\alpha^\epsilon} & \mathcal{O}_{U_\alpha^\epsilon}^{\oplus r}. \end{array}$$

On  $U_\alpha^\epsilon$  where  $U_\alpha^\epsilon \cap \text{Supp}(D_\epsilon(\tilde{\mathbf{t}})) = \emptyset$  and  $U_\alpha^\epsilon \cap \text{Supp}(D_\epsilon(\sigma_M)) \neq \emptyset$ , the relative initial connection  $\nabla_{0,\epsilon}^{\sigma_M, \text{IMD}}$  is described as follows:

$$(36) \quad (\psi_\alpha^\epsilon \otimes \text{id}) \circ (\nabla_{0,\epsilon}^{\sigma_M, \text{IMD}}) \circ (\psi_\alpha^\epsilon)^{-1} = T_\alpha^\epsilon d((T_\alpha^\epsilon)^{-1}) + T_\alpha^\epsilon \text{diag}(-g_\alpha^\epsilon d(g_\alpha^\epsilon)^{-1}, \dots, -g_\alpha^\epsilon d(g_\alpha^\epsilon)^{-1})(T_\alpha^\epsilon)^{-1}.$$

Since the  $\epsilon$ -term of  $a_{k,\alpha}^\epsilon$  vanishes, we have that  $(\psi_\alpha^{-1} \circ \psi_\alpha^\epsilon \otimes \text{id}) \circ (\nabla_{0,\epsilon}^{\sigma_M, \text{IMD}}) \circ (\psi_\alpha^{-1} \circ \psi_\alpha^\epsilon)^{-1} - \tilde{\nabla}_0^{\sigma_M}$  has no pole on the supports of  $D(\sigma_M)$ .

By the argument above, we have the vector bundles  $(\tilde{E}_{\mathcal{C}_{\tau(M)}}^\tau, \{\psi_\alpha\}_\alpha)$  and  $(E_\epsilon^\tau, \{\psi_\alpha^\epsilon\}_\alpha)$ . Since  $(\tilde{E}_{\mathcal{C}_{\tau(M)}}^\tau, \{\psi_\alpha\}_\alpha) \cong (\tilde{E}_{\mathcal{C}_{\tau(M)}}^\tau, \{\phi_\alpha^\tau\}_\alpha)$ , there exist isomorphisms  $t_\alpha: \mathcal{O}_{U_\alpha}^{\oplus r} \rightarrow \mathcal{O}_{U_\alpha}^{\oplus r}$  such that  $t_\alpha \circ \psi_\alpha = \phi_\alpha^\tau$  for any  $\alpha$ . We take a lift  $t_\alpha^\epsilon: \mathcal{O}_{U_\alpha}^{\oplus r} \xrightarrow{\sim} \mathcal{O}_{U_\alpha}^{\oplus r}$  of  $t_\alpha$ . Put  $\hat{\phi}_\alpha^\epsilon := t_\alpha^\epsilon \circ \psi_\alpha^\epsilon$ . We can show that

$$\left( \left( (\phi_\alpha^\tau)^{-1} \circ \hat{\phi}_\alpha^\epsilon \right) \otimes \text{id} \right) \circ (\nabla_{0,\epsilon}^{\sigma_M, \text{IMD}}) \circ \left( (\phi_\alpha^\tau)^{-1} \circ \hat{\phi}_\alpha^\epsilon \right)^{-1} - \tilde{\nabla}_0^{\sigma_M}$$

(where  $\alpha$  satisfies that  $U_\alpha^\epsilon \cap \text{Supp}(D_\epsilon(\tilde{\mathbf{t}})) = \emptyset$ ) has no pole on the supports of  $D(m)$  and  $D(\sigma_M)$ . Since  $(E_\epsilon^\tau, \{\hat{\phi}_\alpha^\epsilon\}_\alpha) \cong (E_\epsilon^\tau, \{(\phi_\alpha^\epsilon)^\tau\}_\alpha)$ , we have  $\chi_\alpha \in \mathcal{A}_{\tilde{E}}(D(\tilde{\mathbf{t}}))(U_\alpha)$  which satisfies

$$\begin{aligned} v_\alpha^{\hat{\mu}^{\nabla_0}} - \left( \left( (\phi_\alpha^{-1} \circ \hat{\phi}_\alpha^\epsilon) \otimes \text{id} \right) \circ (\nabla_{0,\epsilon}^{\sigma_M, \text{IMD}}) \circ \left( \phi_\alpha^{-1} \circ \hat{\phi}_\alpha^\epsilon \right)^{-1} - \tilde{\nabla}_0^{\sigma_M} \right) \\ = \tilde{\nabla}_0^{\sigma_M} \circ (\chi_\alpha - \iota(\tilde{\nabla}_0^{\sigma_M}) \circ \text{symb}_1(\chi_\alpha)) - (\chi_\alpha - \iota(\tilde{\nabla}_0^{\sigma_M}) \circ \text{symb}_1(\chi_\alpha)) \circ \tilde{\nabla}_0^{\sigma_M} + (\chi_\alpha - \iota(\tilde{\nabla}_0^{\sigma_M}) \circ \text{symb}_1(\chi_\alpha)) d\epsilon \end{aligned}$$

for each  $\alpha$  where  $U_\alpha^\epsilon \cap \text{Supp}(D_\epsilon(\tilde{\mathbf{t}})) = \emptyset$ . Since the lift  $\hat{\mu}$  vanishes by the map (27) and  $(C_\epsilon, E_\epsilon)$  is a first-order deformation associated to (29), the element  $\chi_\alpha$  has zero of second order at  $\text{Supp}(D(m) + D(\sigma_M)) \cap U_\alpha$ . Then we obtain that  $\{v_\alpha^{\hat{\mu}^{\nabla_0}}\}$  has no pole on the supports of  $D(m)_{\text{red}}$  and  $D(\sigma_M)$ .

We show that  $[\{u_{\alpha\beta}^{\hat{\mu}^{\nabla_0}}\}, \{-v_\alpha^{\hat{\mu}^{\nabla_0}}\}]$  satisfies the cocycle condition. In fact, we have

$$\begin{aligned} d^0(u_{\alpha\beta}^{\hat{\mu}^{\nabla_0}}) &= \tilde{\phi}_\alpha^{-1} \circ \left( \epsilon \left( d \left( \frac{\widehat{\partial \mu_{\alpha\beta}(\epsilon)}}{\partial \epsilon} \frac{\partial z_\alpha}{\partial f_\alpha} \frac{\tilde{A}_\alpha^0}{z_\alpha} \right) - d \left( \frac{\widehat{\partial \mu_{\alpha\beta}(\epsilon)}}{\partial \epsilon} \frac{1}{f_\alpha} \right) \tilde{A}_\alpha - \frac{\widehat{\partial \mu_{\alpha\beta}(\epsilon)}}{\partial \epsilon} \frac{d(\tilde{A}_\alpha)}{f_\alpha} \right. \right. \\ &\quad \left. \left. + \left( \tilde{A}_\alpha \frac{df_\alpha}{f_\alpha} \right) \frac{\widehat{\partial \mu_{\alpha\beta}(\epsilon)}}{\partial \epsilon} \frac{\partial z_\alpha}{\partial f_\alpha} \frac{\tilde{A}_\alpha^0}{z_\alpha} - \frac{\widehat{\partial \mu_{\alpha\beta}(\epsilon)}}{\partial \epsilon} \frac{\partial z_\alpha}{\partial f_\alpha} \frac{\tilde{A}_\alpha^0}{z_\alpha} \left( \tilde{A}_\alpha \frac{df_\alpha}{f_\alpha} \right) \right) + \left( \frac{\widehat{\partial \mu_{\alpha\beta}(\epsilon)}}{\partial \epsilon} \frac{\partial z_\alpha}{\partial f_\alpha} \frac{\tilde{A}_\alpha^0}{z_\alpha} - \frac{\widehat{\partial \mu_{\alpha\beta}(\epsilon)}}{\partial \epsilon} \frac{\tilde{A}_\alpha}{f_\alpha} \right) d\epsilon \right) \circ \tilde{\phi}_\alpha \\ &= \tilde{\phi}_\alpha^{-1} \circ \left( d \left( \frac{\widehat{\partial \mu_{\alpha\beta}(\epsilon)}}{\partial \epsilon} \frac{\partial z_\alpha}{\partial f_\alpha} \frac{\tilde{A}_\alpha^0}{z_\alpha} - \frac{\widehat{\partial \mu_{\alpha\beta}(\epsilon)}}{\partial \epsilon} \frac{\tilde{A}_\alpha}{f_\alpha} \right) + \left[ \frac{\widehat{\partial \mu_{\alpha\beta}(\epsilon)}}{\partial \epsilon} \frac{\tilde{A}_\alpha}{f_\alpha}, \frac{\partial z_\alpha}{\partial f_\alpha} \frac{\tilde{A}_\alpha^0}{z_\alpha} - \frac{\tilde{A}_\alpha df_\alpha}{f_\alpha} \right] \right. \\ &\quad \left. + \left( \frac{\widehat{\partial \mu_{\alpha\beta}(\epsilon)}}{\partial \epsilon} \frac{\partial z_\alpha}{\partial f_\alpha} \frac{\tilde{A}_\alpha^0}{z_\alpha} - \frac{\widehat{\partial \mu_{\alpha\beta}(\epsilon)}}{\partial \epsilon} \frac{\tilde{A}_\alpha}{f_\alpha} \right) d\epsilon \right) \circ \tilde{\phi}_\alpha \\ &= -(v_\beta^{\hat{\mu}^{\nabla_0}} - v_\alpha^{\hat{\mu}^{\nabla_0}}). \end{aligned}$$

Moreover, we can show the compatibility of  $u_{\alpha\beta}^{\hat{\mu}^{\nabla_0}}$  and  $v_\alpha^{\hat{\mu}^{\nabla_0}}$  with the parabolic structures.  $\square$

**4.3. Hamiltonian structure on the moduli spaces.** In this section, we assume that  $r$  and  $d$  are coprime. Then the relative initial connection  $\tilde{\nabla}_0^{\sigma_M}$  has no pole at  $D(\tilde{\mathbf{t}})$ :

$$\tilde{\nabla}_0^{\sigma_M}: \tilde{E} \longrightarrow \tilde{E} \otimes \Omega_{\mathcal{C}_M/M}^1(D(m)_{\text{red}} + D(\sigma_M)).$$

Let  $M$  be an affine open set of  $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ . We define an algebraic splitting  $\eta: \mathbf{H}^1(\mathcal{G}_M^\bullet) \rightarrow \mathbf{H}^1(\mathcal{F}_M^\bullet)$  of the tangent map  $\mathbf{H}^1(\mathcal{F}_M^\bullet) \rightarrow \mathbf{H}^1(\mathcal{G}_M^\bullet)$  by

$$(37) \quad \eta: \mathbf{H}^1(\mathcal{G}_M^\bullet) \longrightarrow \mathbf{H}^1(\mathcal{F}_M^\bullet); \quad [(\{u_{\alpha\beta}\}, \{v_\alpha\})] \longmapsto [(\{\eta_1(u_{\alpha\beta})\}, \{\eta_2(v_\alpha)\})].$$

Here we set

$$\begin{aligned} \eta_1(s) &:= s - \iota(\nabla) \circ \text{symb}_1(s), \quad s \in \mathcal{G}_M^0 \\ \eta_2: \mathcal{E}nd(\tilde{E}) \otimes \tilde{\Omega}^1 &\longrightarrow \mathcal{E}nd(\tilde{E}) \otimes \Omega_{\mathcal{C} \times_T M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)}^1(D(\tilde{\mathbf{t}})) \end{aligned}$$

which is the projection where

$$\tilde{\Omega}^1 = \Omega_{\mathcal{C} \times_T M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)}^1(D(\tilde{\mathbf{t}})) \oplus \mathcal{O}_{\mathcal{C} \times_T M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d, \nu)} d\epsilon.$$

First, we define a lift of the symplectic form defined in 2.3 as follows. We define a pairing

$$(38) \quad \begin{aligned} &\mathbf{H}^1(\mathcal{C}_M, \mathcal{G}_M^\bullet) \otimes \mathbf{H}^1(\mathcal{C}_M, \mathcal{G}_M^\bullet) \longrightarrow \mathbf{H}^2(\mathcal{C}_M, \Omega_{\mathcal{C}_M/M}^\bullet) \cong H^0(\mathcal{O}_M) \\ &[(\{u_{\alpha\beta}\}, \{v_\alpha\})] \otimes [(\{u'_{\alpha\beta}\}, \{v'_\alpha\})] \longmapsto [(\{\text{Tr}(\eta_1(u_{\alpha\beta}) \circ \eta_1(u'_{\alpha\beta}))\}, -\{\text{Tr}(\eta_1(u_{\alpha\beta}) \circ \eta_2(v'_\alpha)) - \text{Tr}(\eta_2(v_\alpha) \circ \eta_1(u'_{\alpha\beta}))\})], \end{aligned}$$

denoted by  $\omega_M$ , where we consider in Čech cohomology with respect to an affine open covering  $\{U_\alpha\}$  of  $\mathcal{C} \times_T M$ ,  $\{u_{\alpha,\beta}\} \in C^1(\mathcal{G}_M^0)$ ,  $\{v_\alpha\} \in C^0(\mathcal{G}_M^1)$ . By Proposition 3.7 and the construction of  $\omega_M$ , we have the following

**Proposition 4.5.** *The kernel  $\text{Ker}(\omega_M)$  of  $\omega_M: \mathbf{H}^1(\mathcal{G}_M^\bullet) \rightarrow \text{Hom}_{\mathcal{O}_M}(\mathbf{H}^1(\mathcal{G}_M^\bullet), \mathcal{O}_M)$  implies the vector field on  $M$  determined by the isomonodromic deformation.*

Second, we define Hamiltonian functions as follows. The second order poler parts of  $\text{Tr}((\nabla - \tilde{\nabla}_0^{\sigma_M})^2)$  at the support of  $D(\tilde{\mathbf{t}})$  are

$$(39) \quad \left\{ \left( \sum_{j=0}^{r-1} (\nu_j^{(i)})^2 \right) \frac{df_i \otimes df_i}{f_i^2} \right\} \\ \in H^0(\Omega_{\mathcal{C}_M/M}^{\otimes 2}(2D(\tilde{\mathbf{t}}) + 2(D(m)_{\text{red}} + D(\sigma_M))))/\Omega_{\mathcal{C}_M/M}^{\otimes 2}(2(D(m)_{\text{red}} + D(\sigma_M)))).$$

Since the obstruction vanishes, we can take  $Q_0 \in H^0(\Omega_{\mathcal{C}_M/M}^{\otimes 2}(2D(\tilde{\mathbf{t}}) + 2(D(m)_{\text{red}} + D(\sigma_M))))$  such that the image of  $Q_0$  is the element (39). Put

$$(40) \quad Q := \text{Tr}((\nabla - \tilde{\nabla}_0^{\sigma_M})^2) - Q_0 \in H^0(\Omega_{\mathcal{C}_M/M}^{\otimes 2}(D(\tilde{\mathbf{t}}) + 2(D(m)_{\text{red}} + D(\sigma_M)))).$$

We take  $\mu_1, \dots, \mu_{3g-3+n} \in H^1(\Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}})))$  such that  $\{(\mu_1)_x, \dots, (\mu_{3g-3+n})_x\}$  is a basis in  $H^1(\Theta_{\mathcal{C}_x}(-D(\tilde{\mathbf{t}})_{\mathcal{O}_{\mathcal{C}_x}}))$  for any  $x \in M$ . We take lifts  $\hat{\mu}_1, \dots, \hat{\mu}_{3g-3+n} \in H^1(\Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}) - 2(D(m)_{\text{red}} + D(\sigma_M))))$  as (26). We consider the following pairing

$$\begin{aligned} &H^0(\Omega_{\mathcal{C}_M/M}^{\otimes 2}(D(\tilde{\mathbf{t}}) + 2(D(m)_{\text{red}} + D(\sigma_M)))) \otimes H^1(\Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}) - 2(D(m)_{\text{red}} + D(\sigma_M)))) \\ &\longrightarrow H^1(\Omega_{\mathcal{C}_M/M}^1) \cong H^0(\mathcal{O}_M). \end{aligned}$$

**Definition 4.6.** By the pairing, we define *Hamiltonian functions*  $H_i$  ( $i = 1, \dots, 3g-3+n$ ) on  $M$  as

$$(41) \quad H_i = \frac{1}{2} Q \cdot \hat{\mu}_i$$

for the lifts  $\hat{\mu}_1, \dots, \hat{\mu}_{3g-3+n}$ .

By Proposition 4.3, for the lifts  $\hat{\mu}_1, \dots, \hat{\mu}_{3g-3+n}$ , we have the vector fields  $\partial/\partial t_i$  on  $M$ :

$$\frac{\partial}{\partial t_i} := \left[ \left( \left\{ u_{\alpha\beta}^{\hat{\mu}_i \nabla_0} \right\}, \left\{ -v_{\alpha}^{\hat{\mu}_i \nabla_0} \right\} \right) \right] \in \mathbf{H}^1(\mathcal{G}_M^\bullet).$$

**Theorem 4.7.** *Assume that  $r$  and  $d$  are coprime. Let  $d_{M/T} H_i \in \text{Hom}_{\mathcal{O}_M}(\mathbf{H}^1(\mathcal{F}_M^\bullet), \mathcal{O}_M)$  be the relative exterior derivative of  $H_i$ . We define  $dH_i \in \text{Hom}_{\mathcal{O}_M}(\mathbf{H}^1(\mathcal{G}_M^\bullet), \mathcal{O}_M)$  as*

$$dH_i: \mathbf{H}^1(\mathcal{G}_M^\bullet) \longrightarrow \mathcal{O}_M; \quad [(\{u_{\alpha\beta}\}, \{v_\alpha\})] \longmapsto d_{M/T} H_i[(\{\eta_1(u_{\alpha\beta})\}, \{\eta_2(v_\alpha)\})].$$

*Then the one form  $\omega_M(\partial/\partial t_i, \cdot) \in \text{Hom}_{\mathcal{O}_M}(\mathbf{H}^1(\mathcal{G}_M^\bullet), \mathcal{O}_M)$  is coincide with the one form  $dH_i$ .*

*Proof.* Let  $[(\{u'_{\alpha\beta}\}, \{v'_\alpha\})]$  be an element of  $\mathbf{H}^1(\mathcal{G}_M^\bullet)$ , and let  $E'_\epsilon$  be a first-order deformation of  $\tilde{E}_{\mathcal{C}_M}$  associated to  $[(\{\eta_1(u'_{\alpha\beta})\}, \{\eta_2(v'_\alpha)\})] \in \mathbf{H}^1(\mathcal{F}_M^\bullet)$ . We take a first-order deformation  $\tilde{\nabla}_0^{\sigma_M}$  on  $E'_\epsilon$ :

$$(\nabla_{0,\epsilon}^{\sigma_M})': E'_\epsilon \longrightarrow E'_\epsilon \otimes \Omega_{\mathcal{C}_M/M}^1(D(m)_{red} + D(\sigma_M)).$$

Let

$$\varphi'_\alpha: E'_\epsilon|_{U_\alpha \times \text{Spec } \mathcal{O}_M[\epsilon]} \xrightarrow{\sim} \mathcal{O}_{U_\alpha \times \text{Spec } \mathcal{O}_M[\epsilon]}^{\oplus r} \xrightarrow{\sim} \tilde{E}|_{U_\alpha} \otimes \mathcal{O}_M[\epsilon]$$

be an isomorphism such that  $\varphi'_\alpha \otimes \mathcal{O}_M[\epsilon]/(\epsilon): E_\epsilon \otimes \mathcal{O}_M[\epsilon]/(\epsilon)|_{U_\alpha} \xrightarrow{\sim} \tilde{E}|_{U_\alpha} \otimes \mathcal{O}_M[\epsilon]/(\epsilon) = \tilde{E}|_{U_\alpha}$  is the given isomorphism and that  $\varphi'_\alpha|_{t_i \otimes \mathcal{O}_M[\epsilon]}((l_\epsilon)_j^{(i)}) = \tilde{l}_j^{(i)}|_{U_\alpha \times \text{Spec } \mathcal{O}_M[\epsilon]}$  if  $\tilde{t}_i|_{\mathcal{C}_M} \cap U_\alpha \neq \emptyset$ . We put

$$(v_\alpha^0)' := ((\varphi'_\alpha)^{-1} \otimes \text{id}) \circ ((\nabla_{0,\epsilon}^{\sigma_M})' \circ \varphi'_\alpha - \tilde{\nabla}_0^{\sigma_M}) \in \mathcal{E}nd(\tilde{E}) \otimes \Omega_{\mathcal{C}_M/M}^1(D(m)_{red} + D(\sigma_M)).$$

The collection  $(\{\eta_1(u'_{\alpha\beta})\}, \{(v_\alpha^0)'\})$  satisfies

$$(42) \quad \tilde{\nabla}_0^{\sigma_M} \circ (\eta_1(u'_{\alpha\beta})) - (\eta_1(u'_{\alpha\beta})) \circ \tilde{\nabla}_0^{\sigma_M} = (v_\beta^0)' - (v_\alpha^0)'.$$

For  $[(\{\eta_1(u'_{\alpha\beta})\}, \{\eta_2(v'_\alpha)\})] \in \mathbf{H}^1(\mathcal{F}_M^\bullet)$ , we compute  $\omega_M(\partial/\partial t_i, [(\{\eta_1(u'_{\alpha\beta})\}, \{\eta_2(v'_\alpha)\})])$  as follows. We have

$$(43) \quad \begin{aligned} & -\text{Tr}(\eta_1(u_{\alpha\beta}^{\hat{\mu}_i \nabla_0}) \circ \eta_2(v'_\beta)) + \text{Tr}(-\eta_2(v_{\alpha}^{\hat{\mu}_i \nabla_0}) \circ \eta_1(u'_{\alpha\beta})) \\ & = \text{Tr}((- \eta_1(u_{\alpha\beta}^{\hat{\mu}_i \nabla_0})) \circ (\eta_2(v'_\alpha) - (v_\alpha^0)')) \\ & + \left( \text{Tr}(-\eta_1(u_{\alpha\beta}^{\hat{\mu}_i \nabla_0}) \circ (v_\beta^0)') - \text{Tr}(\eta_2(v_{\alpha}^{\hat{\mu}_i \nabla_0}) \circ \eta_1(u'_{\alpha\beta})) \right) - \text{Tr}(\eta_1(u_{\alpha\beta}^{\hat{\mu}_i \nabla_0}) \circ (\nabla_{\mathcal{G}_0^\bullet}(\eta_1(u'_{\alpha\beta})) - \nabla_{\mathcal{G}^\bullet}(\eta_1(u'_{\alpha\beta})))) \end{aligned}$$

where  $\nabla_{\mathcal{G}_0^\bullet}(s) := \tilde{\nabla}_0^{\sigma_M} \circ s - s \circ \tilde{\nabla}_0^{\sigma_M}$  and  $\nabla_{\mathcal{G}^\bullet}(s) := \tilde{\nabla} \circ s - s \circ \tilde{\nabla}$ . We consider the second term of the right hand side of (43). Since

$$\tilde{\nabla}_0^{\sigma_M} \circ (-\eta_1(u_{\alpha\beta}^{\hat{\mu}_i \nabla_0})) - (-\eta_1(u_{\alpha\beta}^{\hat{\mu}_i \nabla_0})) \circ \tilde{\nabla}_0^{\sigma_M} = -\left( \tilde{\nabla} \circ (\eta_1(u_{\alpha\beta}^{\hat{\mu}_i \nabla_0})) - (\eta_1(u_{\alpha\beta}^{\hat{\mu}_i \nabla_0})) \circ \tilde{\nabla} \right),$$

we have

$$(44) \quad \tilde{\nabla}_0^{\sigma_M} \circ (-\eta_1(u_{\alpha\beta}^{\hat{\mu}_i \nabla_0})) - (-\eta_1(u_{\alpha\beta}^{\hat{\mu}_i \nabla_0})) \circ \tilde{\nabla}_0^{\sigma_M} = \eta_2(v_\beta^{\hat{\mu}_i \nabla_0}) - \eta_2(v_\alpha^{\hat{\mu}_i \nabla_0})$$

by the cocycle condition of  $[\{u_{\alpha\beta}^{\hat{\mu}_i \nabla_0}\}, \{-v_\alpha^{\hat{\mu}_i \nabla_0}\}]$ . By the equation (42) and the equation (44), we have that

$$(45) \quad [(-\{\text{Tr}(\eta_1(-u_{\alpha\beta}^{\hat{\mu}_i \nabla_0}) \circ \eta_1(u'_{\beta\gamma}))\}, \{\text{Tr}(-\eta_1(u_{\alpha\beta}^{\hat{\mu}_i \nabla_0}) \circ \eta_2(v_\beta^0)') - \text{Tr}(\eta_2(v_\alpha^{\hat{\mu}_i \nabla_0}) \circ \eta_1(u'_{\alpha\beta}))\})]$$

is an element of  $\mathbf{H}^2(\mathcal{C}_M, \Omega_{\mathcal{C}_M/M}^\bullet)$ .

We claim that the element (45) of  $\mathbf{H}^2(\mathcal{C}_M, \Omega_{\mathcal{C}_M/M}^\bullet)$  vanishes. We can show this vanishing as follows. We can assume that

- (1)  $v_\alpha^{\hat{\mu}_i \nabla_0} = 0$  for any  $\alpha$ , and
- (2)  $(v_\alpha^0)' = 0$  for  $\alpha$  such that  $U_\alpha \cap (\text{Supp} D(m) \cup \text{Supp} D(\sigma_M)) = \emptyset$ .

In fact, since  $r$  and  $d$  are coprime, it is not necessary to consider the elementary transform (23) in the construction of  $\tilde{\nabla}_0^{\sigma_M}$ . Namely, the relative initial connection  $\tilde{\nabla}_0^{\sigma_M}$  has no pole at  $D(\tilde{\mathbf{t}})$ . We consider the class  $[\{u_{\alpha\beta}^{\hat{\mu}_i \nabla_0}\}, \{-v_\alpha^{\hat{\mu}_i \nabla_0}\}]$ . Let  $\{\psi_\alpha^\epsilon\}$  be trivializations of  $E_\epsilon$  defined in the proof of Lemma 4.4. We take a representative of the class  $[\{u_{\alpha\beta}^{\hat{\mu}_i \nabla_0}\}, \{-v_\alpha^{\hat{\mu}_i \nabla_0}\}]$  associated to the trivializations  $\{\psi_\alpha^\epsilon\}$ . Then we can assume that  $v_\alpha^{\hat{\mu}_i \nabla_0} = 0$  for any  $\alpha$ . Next we consider the collection  $(\{\eta_1(u'_{\alpha\beta})\}, \{(v_\alpha^0)'\})$ . We also take trivializations  $\{(\psi_\alpha^\epsilon)'\}$  of  $E'_\epsilon$  as in the proof of Lemma 4.4. By the trivializations  $\{(\psi_\alpha^\epsilon)'\}$ , we have a new collection. We replace  $(\{\eta_1(u'_{\alpha\beta})\}, \{(v_\alpha^0)'\})$  for the new collection. Then we have  $(v_\alpha^0)' = 0$  for  $\alpha$  such that  $U_\alpha \cap (\text{Supp} D(m) \cup \text{Supp} D(\sigma_M)) = \emptyset$ . By these assumptions and the vanishing of images by the morphism (27), we obtain that the element (45) of  $\mathbf{H}^2(\mathcal{C}_M, \Omega_{\mathcal{C}_M/M}^\bullet)$  vanishes.

Next, we consider the third term of the right hand side of (43).

$$\begin{aligned} & \text{Tr}(\eta_1(u_{\alpha\beta}^{\tilde{\mu}_i \nabla_0}) \circ (\nabla_{\mathcal{G}^\bullet} \eta_1(u'_{\alpha\beta}) - \nabla_{\mathcal{G}^\bullet} \eta_1(u'_{\alpha\beta})) \\ &= \text{Tr} \left( \frac{\widehat{\partial \mu_{\alpha\beta}(\epsilon)}}{\partial \epsilon} \left( \frac{\tilde{A}_\alpha^0}{g_\alpha} - \frac{\partial f_\alpha}{\partial g_\alpha} \frac{\tilde{A}_\alpha}{f_\alpha} \right) \circ ((\tilde{\nabla}_0^{\sigma_M} - \tilde{\nabla}) \circ \eta_1(u'_{\alpha\beta}) - \eta_1(u'_{\alpha\beta}) \circ ((\tilde{\nabla}_0^{\sigma_M} - \tilde{\nabla})) \right) = 0. \end{aligned}$$

Then we obtain that

$$\omega(\partial/\partial t_i, [(\{\eta_1(u'_{\alpha\beta})\}, \{\eta_2(v'_\alpha)\})]) = \text{Tr}((- \eta_1(\tilde{u}_{\alpha\beta}^{\tilde{\mu}_i \nabla_0}) \circ (\eta_2(v'_\alpha) - (v'_\alpha)^0)).$$

On the other hand, the one form  $dH_i \in \mathbf{H}^1(\mathcal{G}_M^\bullet)^\vee$  is described as

$$\begin{aligned} dH_i: \mathbf{H}^1(\mathcal{G}_M^\bullet) &\longrightarrow H^0(\mathcal{O}_M) \\ [(\{u'_{\alpha\beta}\}, \{v'_\alpha\})] &\longmapsto \left[ \left\{ \text{Tr} \left( \frac{\widehat{\partial \mu_{\alpha\beta}(\epsilon)}}{\partial \epsilon} \left( \frac{\partial f_\alpha}{\partial g_\alpha} \frac{\tilde{A}_\alpha}{f_\alpha} - \frac{\tilde{A}_\alpha^0}{g_\alpha} \right) (\eta_2(v'_\alpha) - (v'_\alpha)^0) \right) \right\} \right]. \end{aligned}$$

Then we obtain that  $\omega_M(\partial/\partial t_i, [(\{\eta_1(u'_{\alpha\beta})\}, \{\eta_2(v'_\alpha)\})]) = dH_i[(\{u'_{\alpha\beta}\}, \{v'_\alpha\})]$ .  $\square$

Set  $N = r^2(g-1) + nr(r-1)/2 + 1$ . Note that  $\dim M = 2N$ . Let  $\partial/\partial q_i \in \mathbf{R}^1\pi_*(\mathcal{F}^\bullet)(M)$  and  $\partial/\partial p_i \in \mathbf{R}^1\pi_*(\mathcal{F}^\bullet)(M)$  ( $i = 1, \dots, N$ ) be vector fields on  $M$  such that the morphism

$$(46) \quad \begin{aligned} & \mathcal{O}_M^{\oplus 2N} \longrightarrow \mathbf{R}^1\pi_*(\mathcal{F}^\bullet)(M) \\ & (f_1, \dots, f_{2N}) \longmapsto f_1 \partial/\partial q_1 + \dots + f_N \partial/\partial q_N + f_{N+1} \partial/\partial p_1 + \dots + f_{2N} \partial/\partial p_N \end{aligned}$$

gives a trivialization of  $\mathbf{R}^1\pi_*(\mathcal{F}^\bullet)(M)$ , and the vector fields satisfy the conditions  $\omega_M(\partial/\partial q_i, \partial/\partial q_j) = \omega_M(\partial/\partial p_i, \partial/\partial p_j) = 0$  and  $\omega_M(\partial/\partial q_i, \partial/\partial p_j) = \delta_{i,j}$  where  $\delta_{i,j}$  is the Kronecker's symbol. Here we also denote by  $\partial/\partial q_i$  and  $\partial/\partial p_i$  the images of  $\partial/\partial q_i \in \mathbf{R}^1\pi_*(\mathcal{F}^\bullet)(M)$  and  $\partial/\partial p_i \in \mathbf{R}^1\pi_*(\mathcal{F}^\bullet)(M)$  by the tangent morphism

$$\mathbf{R}^1\pi_*(\mathcal{F}^\bullet) \longrightarrow \mathbf{R}^1\pi_*(\mathcal{G}^\bullet).$$

**Corollary 4.8.** *Assume that  $r$  and  $d$  are coprime. If we take vector fields  $\partial/\partial q_i \in \mathbf{R}^1\pi_*(\mathcal{G}^\bullet)(M)$  and  $\partial/\partial p_i \in \mathbf{R}^1\pi_*(\mathcal{G}^\bullet)(M)$  as above, then a vector field determined by the isomonodromic deformation on  $M$  is described as*

$$(47) \quad \frac{\partial}{\partial t_j} - \sum_i \left( dH_j \left( \frac{\partial}{\partial p_i} \right) \frac{\partial}{\partial q_i} - dH_j \left( \frac{\partial}{\partial q_i} \right) \frac{\partial}{\partial p_i} \right)$$

for  $j = 1, \dots, 3g-3+n$ .

*Proof.* Let  $X$  be the vector field (47). We show that  $X \in \text{Ker}(\omega_M)$ , which implies that  $X$  is a vector field determined by the isomonodromic deformation (Proposition 4.5). Note that

$$\begin{aligned} \omega_M(\partial/\partial q_i, \cdot): [(\{u_{\alpha\beta}\}, \{v_\alpha\})] &\longmapsto \omega(\partial/\partial q_i, [(\{\eta_1(u_{\alpha\beta})\}, \{\eta_2(v_\alpha)\})]), \\ \omega_M(\partial/\partial p_i, \cdot): [(\{u_{\alpha\beta}\}, \{v_\alpha\})] &\longmapsto \omega(\partial/\partial p_i, [(\{\eta_1(u_{\alpha\beta})\}, \{\eta_2(v_\alpha)\})]), \\ dH_i: [(\{u_{\alpha\beta}\}, \{v_\alpha\})] &\longmapsto d_{M/T} H_i[(\{\eta_1(u_{\alpha\beta})\}, \{\eta_2(v_\alpha)\})]. \end{aligned}$$

We have

$$dH_j - \sum_i \left( dH_j \left( \frac{\partial}{\partial p_i} \right) \omega_M(\partial/\partial q_i, \cdot) - dH_j \left( \frac{\partial}{\partial q_i} \right) \omega_M(\partial/\partial p_i, \cdot) \right) = 0.$$

By Theorem 4.7, we have

$$\omega_M \left( \frac{\partial}{\partial t_j}, \cdot \right) - \sum_i \left( dH_j \left( \frac{\partial}{\partial p_i} \right) \omega_M(\partial/\partial q_i, \cdot) - dH_j \left( \frac{\partial}{\partial q_i} \right) \omega_M(\partial/\partial p_i, \cdot) \right) = 0.$$

Then we obtain that  $X \in \text{Ker}(\omega_M)$ . By the uniqueness of the isomonodromic deformation for a Kodaira–Spencer class, we obtain this corollary.  $\square$

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